Notes for Remote Presentation:

Game Theory/Fairness Modeling

March 23, 2020

Appealing voting methods:

Winner has largest number of first place votes

Winner can beat all other candidates in a 2-way race

Winner is ranked in a high position on many preference schedules of many voters (points for high rank)

Winner did not get lots of last place votes

Winner had lots of 1st or 2nd place votes

Run-off based on first place votes or last place votes

These methods are:

Plurality

Condorcet

Borda

Anti-plurality

Bucklin

Run-off (sequential run-off) and Coombs

Turn the problem on its head:

Instead of searching for appealing methods:

What "good properties" should an excellent voting method obey?

Good voting procedures should obey common sense fairness rules.

Example of a fairness rule:

(Monotonicity:) Getting more support should not harm a candidate.

More fairness rules:

1. No dictator

One person whose opinion is what society chooses to do.

2. Whatever ballots are submitted are used to decide a winner. (The decision process does not "disregard" the ballots - a so-called imposed winner.)

Example of using an imposed method:

Consult the Delphi Oracle

3. Independence of irrelevant alternatives.

The relative position for society of choice X and choice Y should not depend on whether or not choice Z is available or not.

4. If all the voters in society rank candidate X first, society should rank this candidate first.

Kenneth Arrow's Theorem

(Arrow was an undergraduate at City College, now part of CUNY and earned his doctorate degree at Columbia University in economics under the direction of the mathematician statistician Harold Hoteling.)

(Many variants for "Arrow's Theorem")

There is ***no*** election decision method that involves three or more candidates which obeys a short list of fairness rules:

a. Decisive (some winner is found)
b. Not imposed
c. Not dictatorial
d. Obeys independence of irrelevant alternatives.
e. Monotone

Technically, Arrow's work is based on finding a ranking with ties allowed based on ordinal (ranked) ballots with ties allowed.

Elections decided by "grading" - cardinal rather than ordinal ballot

Score voting or range voting

Ballot: give the candidate a grade from 0 to 9

Winner has the largest sum of grades.

Majority Judgment: (Balinski and Laraki)

Give each candidate a grade. Winner has the highest median grade.

Lots of ties are common and many variants as to how to break them.

Arrow's work does not "apply" to these systems.

However, the next result, due to Alan Gibbard (philosopher) and MarkSatterthwaite (economics/business) applies to election systems with 3 or more choices, using any kind of ballot and election decision method - except one.

The Borda Count seems like an appealing method though it does not obey Independence of Irrelevant Alternatives.

One obvious criticism: Suppose a voter knows the election will be close between his/her first and second place choices in an election when the voter provides honest preferences. By being honest, the voters second place choice may "harm" his/her first place choice.

So a voter might be tempted to not vote honestly, but put his/her second place choice towards the bottom to cut down on that candidate's points. However, all voters may vote dishonestly resulting in an outcome not at all in the interests of the group.

Such voting behavior is called "*strategic*" *voting* or tactical voting. Strategic voters use information about the choice of voting system and "information" (typically from polls) about how other voters may vote, to take actions which help their favorite candidates even though they are "lying" about their true opinions.

Satterethwaite-Gibbard Theorem:

When there are at least three choices the only election method which is immune from strategic voting (manipulation) is *dictatorship*!

Many people feel that commentary on elections on the night of voting is unfair because states further west have voting going on after the polls on the East Coast have closed.

Some countries prohibit this practice as well as polls taken close to the time of the election.

Arrow's Theorem is sometimes summarized by saying there is no perfect voting system when there are at least 3 candidates.

However, this does not mean we can't do better than Plurality, which is almost universally used. Moving to ranked ballots is a first step.

Since every improvement over plurality violates some fairness condition, there is constant debate about what method to move to. We will see if the voters in NYC are happier with IRV (sequential run-off) when it comes into use in 2021.

Voting and Weighted voting games:

Electoral College (US)

European Union

NY State County government

Amending the Canadian Constitution

UN Security Council

Weighted voting:

Basic idea - players don't cast equal number of votes because they represent political entities of unequal sizes.

Example:

[5; 4, 3, 2] Three players named 1, 2, and 3 who cast 4, 3, and 2 votes respectively. The 5 is called the quota. Players with combined weight of 5 are needed to take action.

Is Player 1 twice as powerful as Player 3 because 4 is twice 2?

[5; 4, 3, 2]

Which coalitions (collections) of players can take action?

Minimal winning coalitions - no subset of a minimal winning (MW) coalition wins:

{1,2}, {1,3}, {2,3}

Given [5; 4, 3, 2], we have total symmetry here for the MW. The MW coalitions are:

{1,2}, {1,3}, {2,3}

so it should be apparent that in this game all three players have equal influence!!!

An isomorphic game would be:

[2; 1, 1, 1]

because its minimal winning coalitions are also:

$$\{1,2\}, \{1,3\}, \{2,3\}$$

Power indices: (Variants differ in using all winning versus MW coalitions)

a. Coleman

b. Banzhaf

c. Shapely

d. Deegan-Packel-Johnston

[5; 4, 3, 2]

MW: {1,2}, {1,3}, {2,3}

Coleman:

1 is in two coalitions
2 is in two coalitions
3 is in two coalitions

So 1 has 2/6 as a power; 2 has 2/6 as a power; 3 has 2/6 as a power!

Look at the pattern of Yes and No votes of the 3 players:

YYY wins
$\underline{\text{YY}}$N wins
$\underline{\text{Y}}$N$\underline{\text{Y}}$ wins
YNN loses
N$\underline{\text{YY}}$ wins
NYN loses
NNY loses
NNN loses

Underlines show when a Yes changed to a No changes a win to a loss. So of the underlined items each player has 2 out of a total of 6. (This is Banzhaf Power.)

Shapley-Shubik Power Index

[5; 4, 3, 2]

1	*2*	3
1	*3*	2
2	*1*	3
2	*3*	1
3	*1*	2
3	*2*	1

Pivot player is shown in italics - second in every case for this example.

Hence:

Player 1 has 2 pivots out of 6; power 1/3

Player 2 has 2 pivots out of 6; power 1/3

Player 3 has 2 pivots out of 6; power 1/3

Remember that 2/6 is the same fraction as 1/3.

Apportionment:

Here are two "not realistic" but instructive examples to try?

Three Claimants:

h = 12 (Size of the legislature)

A	B	C	
43	36	21	(Total 100)

How many seats for each claimant?

Four Claimants:

h = 12 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

What is the algorithm or method behind what you did so that you can do this for other examples with different numbers of claimants, population sizes and where h varies?

Four Claimants:

h = 12 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

Students typically discover for themselves (without any prior knowledge of this circle of ideas) two methods. Let us see how this might happen.

Some natural thoughts:

There being 12 seats (h=12) what is the exact share each claimed should get?

A has 38 percent of the population. Hence, .38(12)

is A's fair share!

A is entitled to 4.56 seats but this is not an integer.

Fair share of all the claimants (which sum to 12)

A: 4.56
B: 3.24
C: 2.76
D: 1.44

Another view of fair share:

100 people are to be represented by 12 people:

100/12 = 8.33333....

So A's number of seats should be: 38/8.3333=4.56

B's share: 27/8.3333=3.24

C's share: 23/8.3333=2.76

D's share: 12/8.333 =1.44

By the laws of arithmetic (!) these are the same numbers we got before!

If claimant A's population is p(A) and the total populations is T we have:

$$(P(A)/T)h = P(A)/(T/h)$$

Hamilton's Method:

Step 1:

We assign each claimant the integer part of their claim.

Step 2:

If these numbers don't add to 12, order the fractional parts in order of decreasing size and assign these in order of size until

12 seats are assigned.
In this example:

A gets 4 + 1 = 5
B gets 3 = 3
C gets 2 + 1 = 3
D gets 1 = 1

Total 12 as required. Hamilton's method is easy to carry out and intuitive. However, it has two big flaws:

Hamilton's Method fails "monotonicity." If can give out fewer seats to a claimant when the house size goes from h to h+1.

This is not serious in apportioning the House of Representatives because the number of seats stays 435.

Historical anomaly when Hawaii and Alaska became states.

More serious: Hamilton's Method is not population monotone. Various versions of this "axiom." Intuitively, a state could go up in population but get fewer seats.

This can happen with examples involving absolute growth of population or relative growth of population.

Another intuitive approach to dealing with the fractions: when fair share is not an integer, round using the usual rounding rule.

Fair share of all the claimants (which sum to 12)

A: 4.56 assign A 5 seats
B: 3.24 assign B 3 seats
C: 2.76 assign C 3 seats
D: 1.44 assign D 1 seat

Since we have assigned exactly 12 seats were are done.

Here when we us "usual rounding" we gave away exactly the right number of seats. But does this always happen?

Answer: NO!

Examples:

Four Claimants:

$h = 13$ (Size of the legislature)

A	B	C	D	
38	27	23	12	(Total 100)

How many seats for each claimant?

A's fair share: 4.94
B's fair share: 3.51
C's fair share: 2.99
D's fair share: 1.56

Giving A=5, B=4, C=3, D=2 apportions too many seats!

Four Claimants:

h = 19 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant? (Quota: 5.26)

A's fair share: 7.22
B's fair share: 5.13
C's fair share: 4.37
D's fair share: 2.28

Giving A=7, B=5, C=4, D=2 apportions too few seats!

What is to be done?

Define EXACT QUOTA to be:

(Total population)/h

One can adjust this number up or down (to get an adjusted quota) depending on circumstances to assign exactly h seats after the rounding is carried out!

Four Claimants:

h = 19 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

If we divide by 5.1 instead of 5.26 we get:

A: 38/5.1=7.45 so 7

B: 27/5.1=5.29 so 5

C: 23/5.1 =4.51 so 5 (round up)

D: 12/5.1=2.35 so 2.

Total seats is 19 as needed!

Summary:

a: Hamilton's Method or the Method of Largest Remainders

Assign integer part of fair share and if more seats must be given out do this in order of the size of remainders.

b. Webster's Method (Sainte-Laguë)

Use ordinary rounding. If h seats are assigned, stop. Otherwise by trial and error adjust Exact Quota up or down to assign h seats.

Other important methods are:

Jefferson/D'Hondt

Huntington/Hill (used for the US to Apportion the House of Representatives)

Webster's Method is an example of a divisor method, based on usual rounding.

Other apportionment methods use a different rounding rule:

Jefferson (D'Hondt): Always round down

Adams: Always round up

Dean: Round based on the harmonic mean

Huntington/Hill (Currently the method used in America): Round using the geometry mean

Geometric mean of a and b equals square root (ab)

How does one decide which of these methods is better or worse?

What fairness properties do they obey?