Notes for Remote Presentation:

Game Theory/Fairness Modeling

March 30, 2020

Loose ends:

Strategic voting when plurality is the voting method:

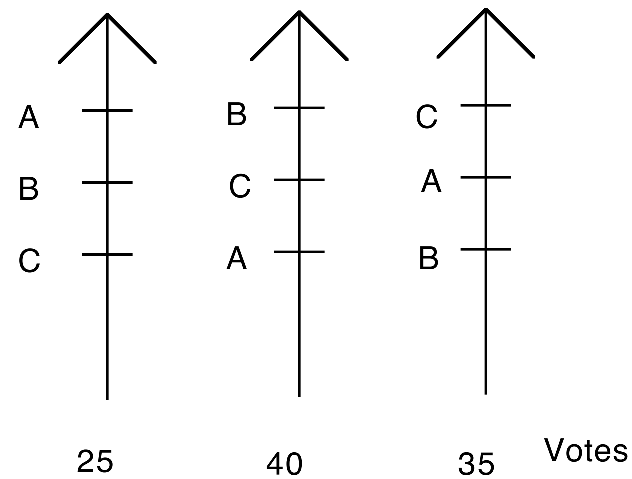
US election for President with candidates Nader, Gore and Bush (2000):

Knowing that Nader had little chance of winning those who truly favored him may have voted strategically for their second choice Gore, as their first choice!

The fact that plurality voting is used in America makes it hard for more than two parties to thrive!

(Nader ran on the Green Party ticket in 2000 as did Jill Stein in 2016.)

Independence of Irrelevant Alternatives and the Borda Count. (Note: No Condorcet winner.)



Borda counts: A gets 85; B gets 105; C gets 110 points.

B gets a higher Borda Count than A when C is present. But without C, 60 to 40 voters prefer A to B. So choice C is not irrelevant when using the Borda Count for the relative position of A and B.

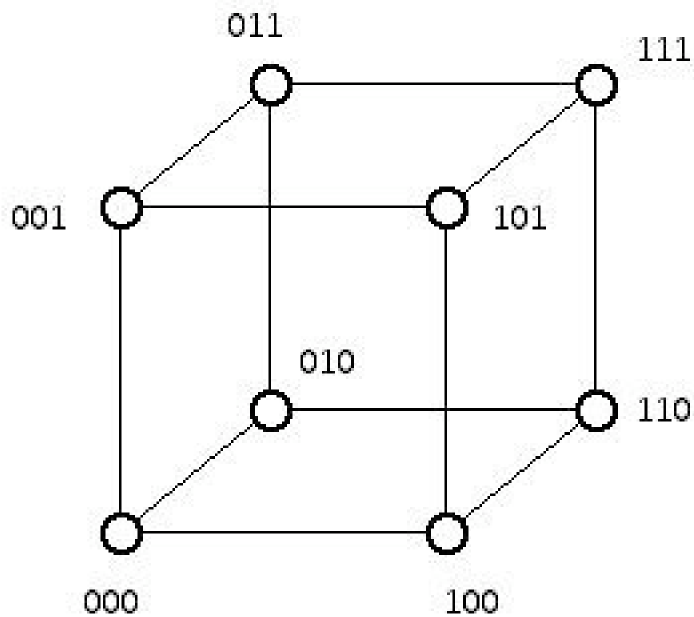
Power indices:

Pattern of Yes/No for lines in a Banzhaf power table for 3-players "corresponds" to the labels needed for a 3-dimensional cube:

NNY,YNY,NYY,YYY(top)

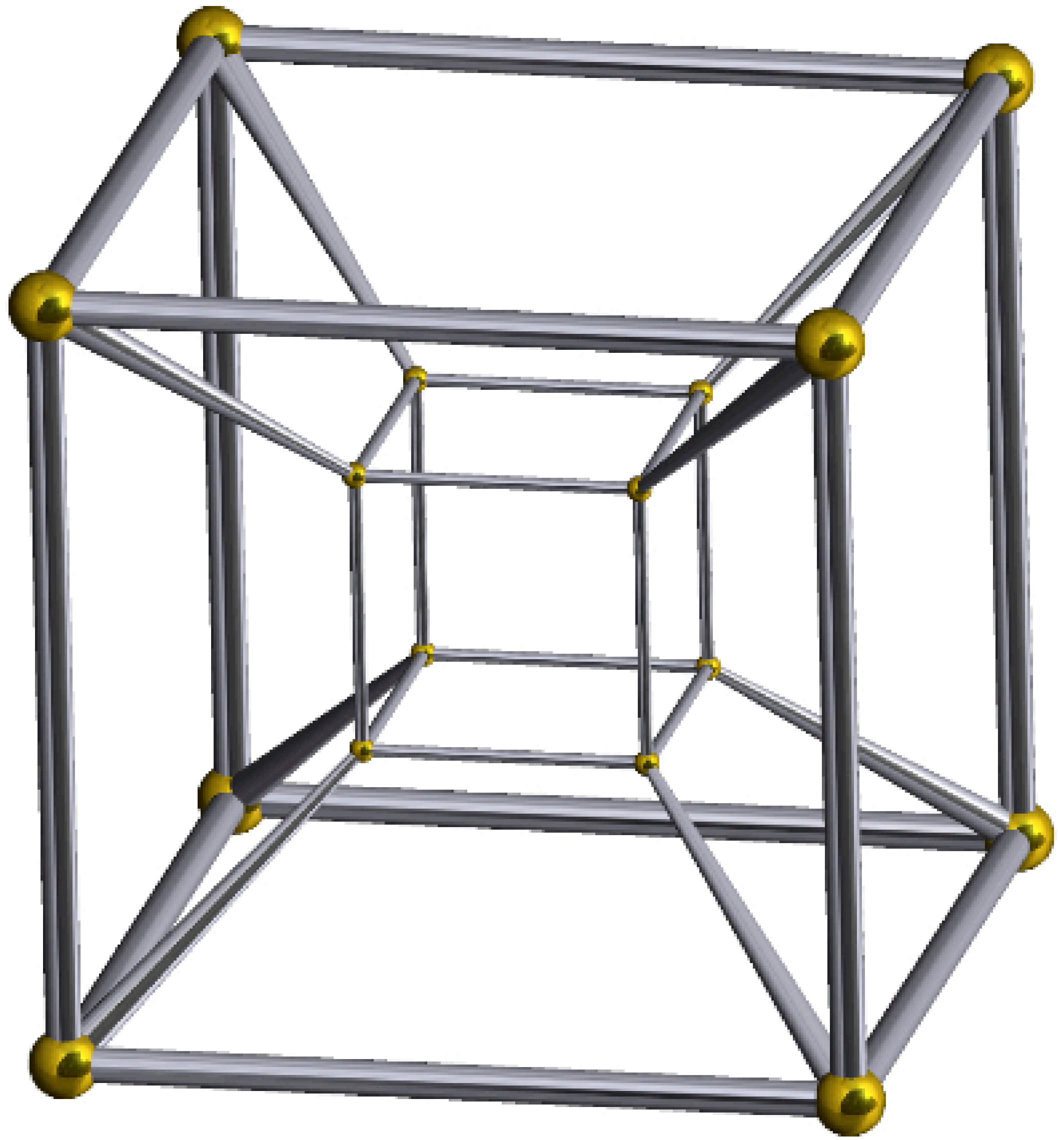
NNN,YNN,NYN,YYN(bottom)

Think of N as a 0 and Y as a 1:



Three-cube made from two 2-cubes! Top layer all entries end in 1; bottom layer all entries end in 0!

Banzhaf table for 4 players correspond is obtained by pasting together two copies of a 3-cube to get a combinatorial 4-cube.

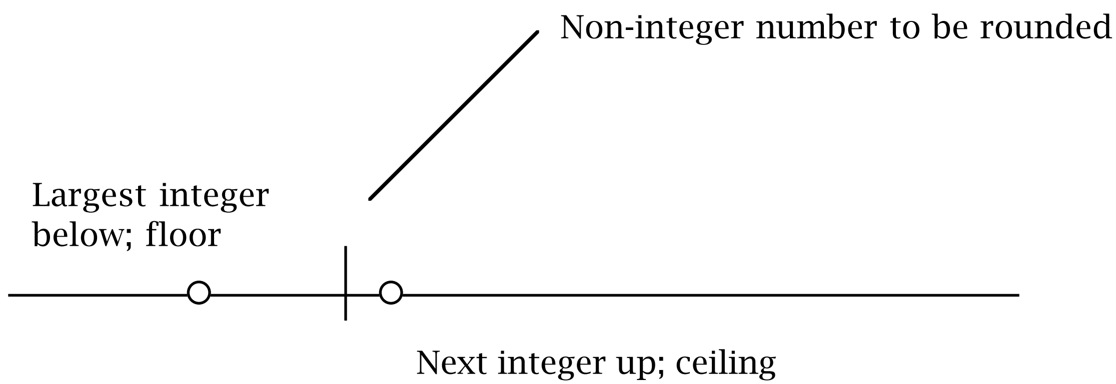


Setting the stage for learning about apportionment.

Classic problem: How many seats does each state get in the House of Representatives based on its population?

Notation and its power:

Rounding:



Notation for floor and ceiling due to Donald Knuth

Floor function:

Largest integer smaller than or equal to x:

$$\lfloor x \rfloor$$

Ceiling function:

Smallest integer greater than or equal to x:

$$\lceil x \rceil$$

Examples:

$\lceil 3.01 \rceil = 4$

$\lfloor 6.72 \rfloor = 6$

Graph: $y = \lceil x^2 \rceil - (\lfloor x \rfloor)^2$

Practice:

$-\lfloor 6.72 \rfloor =$

$\lfloor -6.72 \rfloor =$

$\lceil -12.47 \rceil =$

Lots of interesting graphing examples can be done with this notation in a precalculus class in college or an algebra course in high school.

Absolute versus relative change:

A's population goes from

50,000 to 60,000

B's population goes from

5,000,000 to 5,010,000

Both A and B went up the same amount in "absolute" terms: 10,000 people.

But, 10000/50000 is 20%

while 10000/5000000 is .2%

Apportionment:

Here are two "not realistic" but instructive examples to try.

Three Claimants:

$h = 12$ (Size of the legislature)

A	B	C	
43	36	21	(Total 100)

How many seats for each claimant?

Four Claimants:

h = 12 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

What is the algorithm or method behind what you did so that you can do this for other examples with different numbers of claimants, population sizes and where h varies?

Four Claimants:

h = 12 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

Students typically discover for themselves (without any prior knowledge of this circle of ideas) two methods. Let us see how this might happen.

Some natural thoughts:

There being 12 seats ($h=12$) what is the exact share each claimant should get?

A has 38 percent of the population. Hence, .38(12)

is A's fair share!

A is entitled to 4.56 seats but this is not an integer.

Fair share of all the claimants (which sum to 12)

A: 4.56
B: 3.24
C: 2.76
D: 1.44

Another view of fair share:

100 people are to be represented by 12 people:

100/12 = 8.33333....

Each representative should represent 8.333333.... people.

So A's number of seats (representatives) should be:

A's share: 38/8.3333=4.56
B's share: 27/8.3333=3.24
C's share: 23/8.3333=2.76
D's share: 12/8.333 =1.44

By the laws of arithmetic (!) these are the same numbers we got before!

If claimant A's population is p(A) and the total populations is T we have:

$$(P(A)/T)h = P(A)/(T/h)$$

Hamilton's Method:

Step 1:

We assign each claimant the integer part of their claim.

Step 2:

If these numbers don't add to 12, order the fractional parts in order of decreasing size and assign these in order of size until 12 seats are assigned.

In this example:

A gets 4 + 1 = 5
B gets 3 = 3
C gets 2 + 1 = 3
D gets 1 = 1

Total 12 as required.

Hamilton's method is easy to carry out and intuitive. However, it has two big flaws:

Hamilton's Method fails "monotonicity." If can give out fewer seats to a claimant when the house size goes from h to h+1.

This is not serious in apportioning the House of Representatives because the number of seats stays 435.

Historical anomaly when Hawaii and Alaska became states.

More serious: Hamilton's Method is not population monotone. Various versions of this "axiom."

Intuitively, a state A could go up in population in comparison with another state B, but A would lose a seat to B.

This can happen with examples involving absolute growth of population or relative growth of population.

Another intuitive approach to dealing with the fractions: when fair share is not an integer, is to round using the usual rounding rule.

Fair share of all the claimants (which sum to 12)

A: 4.56 assign A 5 seats
B: 3.24 assign B 3 seats
C: 2.76 assign C 3 seats
D: 1.44 assign D 1 seat

Since we have assigned exactly 12 seats were are done.

Here when we use "usual rounding" we gave away exactly the right number of seats. But does this always happen?

Answer: NO!

Examples:

Four Claimants:

h = 13 (Size of the legislature)

A	B	C	D	
38	27	23	12	(Total 100)

How many seats for each claimant?

A's fair share: 4.94
B's fair share: 3.51
C's fair share: 2.99
D's fair share: 1.56

Giving A=5, B=4, C=3, D=2 apportions too many seats!

Four Claimants:

h = 19 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant? (Quota: 5.26)

A's fair share: 7.22
B's fair share: 5.13
C's fair share: 4.37
D's fair share: 2.28

Giving A=7, B=5, C=4, D=2 apportions too few seats!

What is to be done?

Define EXACT QUOTA to be:

(Total population)/h

One can adjust this number up or down (to get an adjusted quota) depending on circumstances to assign exactly h seats after the rounding is carried out!

Four Claimants:

h = 19 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

If we divide by 5.1 instead of 5.26 we get:

A: 38/5.1=7.45 so 7
B: 27/5.1=5.29 so 5
C: 23/5.1 =4.51 so 5 (round up)
D: 12/5.1=2.35 so 2.
Total seats is 19 as needed!

Summary:

a: Hamilton's Method or the Method of Largest Remainders

Assign integer part of fair share and if more seats must be given out do this in order of the size of remainders.

b. Webster's Method (Sainte-Laguë)

Use ordinary rounding. If h seats are assigned, stop. Otherwise by trial and error adjust Exact Quota up or down to assign exactly h seats.

Other important methods are:

Jefferson/D'Hondt

Huntington/Hill (used for the US to Apportion the House of Representatives)

Webster's Method is an example of a divisor method, based on the usual rounding rule.

We will see an easier approach to finding divisor method apportionments later.

This approach is due to E.V. Huntington and I refer to it as the "table method" approach.

The more formal name for this other approach is rank index methods.

Webster Violates Quota!

A state may get more or less than its fair share rounded up or down to the nearest integer!

1,593,436/35 = 45526.743 assigns too many seats.

Table uses: 46842 to assign 35 seats.

Party (State)	Vote (population)	Exact quota (share) of house	Webster or Huntington-HIll number of seats
A	70,653	1.552	2
B	117,404	2.579	3
C	210,923	4.633	5
D	1,194,456	26.236	25
Total	1,593,436	35	35

Other apportionment methods use a different rounding rule:

Jefferson (D'Hondt): Always round down

Adams: Always round up

Dean: Round based on the harmonic mean

Huntington/Hill (Currently the method used in America): Round using the geometry mean

Geometric mean of a and b equals square root (ab)

Geometric mean of a and b:

$$\sqrt{ab}$$

Geometric mean of a and (a+1):

$$\sqrt{a(a+1)}$$

I drove 200 miles from NYC at 36 mph and made the return trip at 48 miles mph? What was my average speed for the whole trip?

No, the answer is not 42 mph!

It is:

41.142857.......

Harmonic mean of a and b:

$$\frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Harmonic mean of a and (a+1):

$$\frac{2}{\frac{1}{a}+\frac{1}{a+1}}$$

$$\frac{2}{\frac{1}{a}+\frac{1}{a+1}}=\frac{2a(a+1)}{2a+1}$$

How does one decide which of these methods is better or worse?

What fairness properties do they obey?

Fairness axioms:

1. House size h monotonicity.

For many practical problems not important and there are algorithms that obey house size monotonicity and other critical fairness properties.

2. Population monotonicity

3. Quota

A state should get its fair share if it is an integer and the fair share rounded up or down to the next integer if it is not an integer.

Balinski-Young Theorem:

Michel Balinski (1933-2019)
H.P. (Peyton) Young

(who both taught for many years at CUNY's Graduate Center)

There is no apportionment method which obeys both population monotonicity and quota!!