Notes for Remote Presentation:

Game Theory/Fairness Modeling

April 13, 2020

How is the table method used?

The easiest table method to carry out is for Jefferson/D'Hondt, associated with rounding down.

To keep the arithmetic fairly simple I will use numbers that add up to 1000 and have many small prime factors!

A = 400; B = 300; C = 240; D = 60

	A	B	C	D
Original data				
1. Divide by 1				
2. Divide by 2				
3. Divide by 3				
4. Divide by 4				
5. Divide by 5				
6. Divide by 6				
7. Divide by 7				

What might make one apportionment method better than another one?

Think of the different methods as optimizing some "quantity," and pick the method that does this?

Some notation:

Σ means sum; h = house size

Population state i: p_i

T = Total population = Σp_i

Seats assigned to state i: a_i (integer)

State i's fair share: $q_i = ((p_i)/T)h$

(sometimes called state i's quota)

Theorem: Hamilton's Method minimizes:

$$\Sigma\ |a_i - q_i|$$

and also:

$$\Sigma(a_i - q_i)^2$$

What one has done here is to set up a measure of comparison between different apportionments and found the method that with this measure is optimal.

This theorem looks at optimization at a global level.

With other objective functions for optimality, other apportionment emerge as optimal.

Huntington pioneered the use of a different approach to choosing among different apportionments.

His idea was to look at a pair of states and see if society was "happier" with the result when one transferred one seat from one state to the other!

When looking at whether a seat transfer improves things or makes things worse, one needs a way to measure how good things are in the current situation.

One can compare seat transfers between states from either a relative or absolute change point of view.

Reminder:

The relative change between two *positive* quantities a and b is given by:

$$(|a-b|)/\min(a, b)$$

while absolute change is given by:

$$|a-b|$$

The absolute value of a number is always positive.

$|-1-6| = 7$

$|-1+6| = 5$

$|6-1| = 5$

$|-6-1| = 7$

One could, but we will not, look at relative increase and/or decrease of two positive quantities.

Two natural ways to think about what a state received in an apportionment are:

representatives per person:

$$a_i/p_i$$

people per representative:

$$p_i/a_i$$

Now we can measure "disparity" between pairs of states in different ways:

Absolute difference between states j and i:

$$|a_j/p_j - a_i/p_i|$$

or

$$|p_j/a_j - p_i/a_i|$$

Relative difference between states j and i:

$$(|a_j/p_j - a_i/p_i|)/\min(a_j/p_j,\ a_i/p_i)$$

or

$$(|p_j/a_j - p_i/a_i|)/\min(p_j/a_j,\ p_i/a_i)$$

Perhaps unintuitively, there are usually DIFFERENT apportionments that MUST be chosen if you want to be fair with these different "measures" of fairness, even though a_j/p_j and p_j/a_j are reciprocals of each other!

Webster is the method which is best measured by:

$$|a_j/p_j - a_i/p_i|$$

but when one relativizes this:

$$(|a_j/p_j - a_i/p_i|)/\min(a_j/p_j, a_i/p_i)$$

Huntington-Hill is the best choice!!

What are the pros and cons of these different methods? Isn't it somewhat arbitrary which of the 5 rank index (rounding rule equivalents) methods that picks?

Yes and no! But feelings run high.

Typically a party or state endorses that method which maximizes its number of seats - rarely is fairness the issue!

Fact: If one uses a RELATIVE measure of fairness in comparing two states with regard to who get the next seat, then Huntington proved that Huntington-Hill is *mandatory* in all cases.

For the measure

$$| a_i - a_j (P_i / P_j) |$$

Adams method is optimal.

For the measure

$$| P_j / a_j - P_i / a_i |$$

Dean's method is optimal.

For the measure

$$\left| \frac{a_i / P_i}{a_j / P_j} - 1 \right|$$

Huntington-Hill is optimal.

For the measure

$$| a_i / P_i - a_j / P_j |$$

Webster is optimal.

For the measure

$$| a_i (P_j / P_i) - a_j |$$

Jefferson's method is optimal.

However, in addition to the issue of whether one should measure fairness in absolute or relative terms, there is another important idea to discuss.

Referred to as BIAS.

For a single instance of an apportionment problem many times the 5 methods agree, but sometimes they differ. However, perhaps when the same method is used over and over again for MANY apportionment instances there is a systematic way that one method dis "biased" against states in some way, in particular, due to their population size!

Perhaps one method is BIASED for SMALL POPULATION states or favors LARGE POPULATION states.

Looked at in terms of rounding for numbers which lie between consecutive integers a and (a+1) we have this ordering:

Generous in giving an extra seat for a fraction - this is the ordering:

Adams
Dean
Huntington-Hill
Webster
Jefferson

There is wide agreement that Adams consistently rewards smaller states with more seats than they "deserve," while Jefferson (D'Hondt" rewards large states with more seats than they "deserve."

Things get heated between what happens for Webster and Huntington-Hall.

The two leading experts on apportionment Balinski and Young argue that Webster is less biased than Huntington-Hill. Others are not sure. In the US apportionment problem the US constitution is biased towards small states, in the sense that one must give each state one seat no matter how small its population!

I drove 200 miles from NYC at 36 mph and made the return trip at 48 miles mph? What was my average speed for the whole trip?

No, the answer is not 42 mph!

It is:

41.142857.......

Harmonic mean of a and b:

$$\frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Harmonic mean of a and (a+1):

$$\frac{2}{\frac{1}{a}+\frac{1}{a+1}}$$

$$\frac{2}{\frac{1}{a}+\frac{1}{a+1}}=\frac{2a(a+1)}{2a+1}$$

(Dean's method is based on the harmonic mean; never used in America.)

In general the harmonic mean of a and a+1 is smaller than the geometric mean of a and a+1, which in turn is small than the arithmetic mean of a and a+1!

The US Supreme Court has decided a variety of apportionment cases.

In the most important they decided not to agree when Montana went from having 2 seats in the House of Representatives to only 1 seat, that Dean's Method should be used because with Dean's Method Montana would have gotten 2 seats!

Other cases have dealt with the fact that Congress has given over the "mechanics" of carrying out the apportionment to the Commerce Department (which carries out the US Census) and that it use Huntington-Hill.

In the European context of apportionment problems, countries that use D'Hondt method tend to have more "stability" than those that use some of the other methods. Why? The party with the largest vote may get an "assist" by rewarding it with "more seats," which makes it easier to form stable coalitions in parliament. Not all scholars agree!

Similarity some of the kinds of problems we have looked at:

Some pot of items or thing is to be given out to "players" based on their claims.

Problems differ in that the objects distributed may or may not be identical, may or may not be subdividable, must be distributed in integer numbers, etc.

Bankruptcy model:

The origins of the bankruptcy model are many but include both recent and ancient ancestors:

a. A parent has willed various amounts in his/her estate but the size of the estate E is not large enough to cover all the amounts!

b. A company has gone bankrupt but there are not enough remaining assets to pay all of the persons/companies owed money.

Abstract version:

We have claimants (players) 1, 2,...,k whose claims $c_1,...,c_k$ sum to more than a positive amount E. How much should be given to each claimant?

I usually number the players so that claims are decreasing in size.

Comment: Some claimants may be "richer" than others and if they get a small part of their claims they will not be "hurt" badly. Some claimants may go "bankrupt" themselves if they don't get back what they are owed!

This model ONLY looks at the size of the claims and not the "affluence" of the claimants.

The estate E is usually **money** but it could be:

a. water

b. an amount of a limited medical supply

c. time (access to a time share)

d. raising taxes from different income classes (many poor; few rich)

So a typical example of such a bankruptcy problem would like like:

A	B			E = 120
60	90			

A	B	C		E = 150
60	90	120		

A	B	C	D	E = 180
60	90	120	120	

How might one distribute the estate E to the claimants in each of these cases?

Note: Collectively the players get back all of the estate E and we can find the "loss" they collectively incur by subtracting the estate E from the total amount claimed.

Example: A owed 50, B owed 150

$E = 160$; so the loss incurred is 40. (Loss equals $(50 + 150) - 160 = 40$

What fairness ideas do you use?

A	B	E = 120
40	160	

There are surprisingly many appealing ways to decide how to settle the claims!

Common approaches:

1. Entity equity

Treat all claimants alike no matter what the size of their claims.

(Example of entity equity: US Senate - each state gets one seat regardless of its population.)

What happens when you use entity equity in this example?

A	B	E = 120
40	160	

Answer: Give each player 60.

Many feel it is not "fair" to give a claimant more than they asked for!

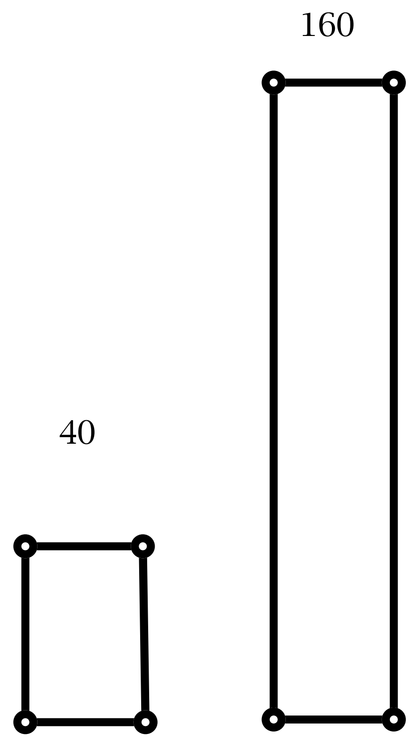
How to modify entity equity to accommodate not giving a claimant more than he/she asked for goes back to Maimonides (1138-1204)

2. (Maimonides gain) Equalize as much as possible the amount given to each claimant but never give a claimant more than he/she asked for!

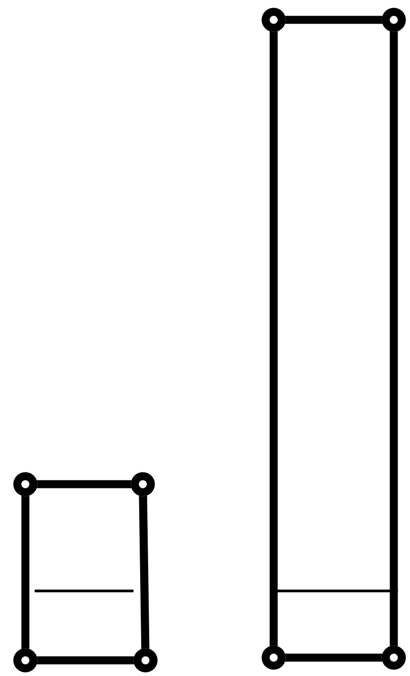
This is what is known as a constrained optimization problem. Can be solved using mathematical programming tools.

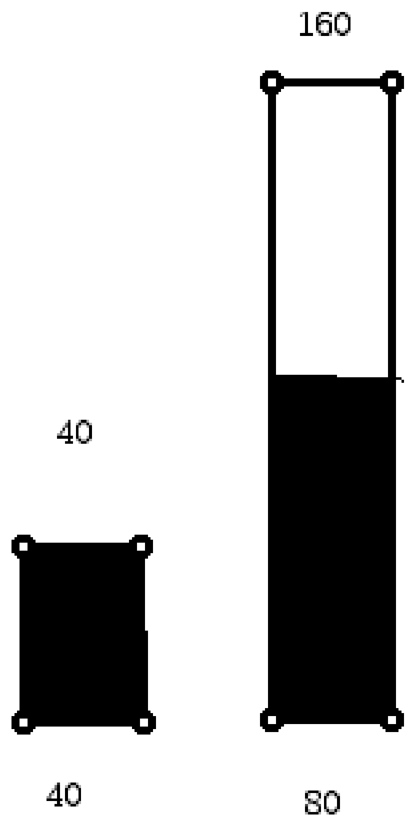
One way to think of solving this kind of problem for two or more claimants is to think of the claims as bins and use a pitcher of fluid with the amount of fluid equal to the size of the estate to fill the bins. One can't pour more fluid into a bin than its capacity! When the pitcher runs out of fluid one is done.

Bins to help solve a claims problem with claims of 40 and 160.



This diagram illustrated how one tries to equalize payments until some claimant(s) gets all of its (their) claim(s) before the estate runs out:





E=120

(final amounts to claimants at the bottom)

3. Equalize losses between the claimants if this means having some claimant subsidize the settlement.

Does not seem fair to many people.

4. Maimonides loss

Equalize losses as much as possible without forcing any claimant to subsidize the settlement.

5. Proportionality

Give each claimant the portion of E that arises from its percentage of the total claims.

6. Concede and divide (sometimes called the contested garment rule and sometimes the Talmudic Method).

Example:

A	B	E = 120
40	160	

B says to the person distributing the estate: A is asking for 40 of 120, so 80 of the money must be mine! This is B's contested claim against A.

In this example A has no uncontested claim against B, but it could happen both have uncontested claim.

Method: give each player their uncontested claim and slit the remainder left in the estate equally!

A	B	E = 120
40	160	

So B gets 80 + 20 = 100

and A gets 0 + 20 = 20

40 units are split equally.

Final: A gets 20 and B gets 100.

6. Shapley value

7. Fixed percent on the dollar, this percentage chosen to use up the estate. (Gives same answer as proportionality.)

Our example:

$40x + 160x = 120$; $x = .6$ so give $A=24$ and $B=96$.