Beyond Plurality Voting

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Introduction

What makes democracies special is that they are lead by representatives who have been selected, that is, elected by the people, not the military or some charismatic single person (dictator). But elections went on during the period Hitler led Germany and Stalin led the Soviet Union. Here I will survey different ways that democracies have organized selecting leaders using elections with the intent of reflecting the views of the people who make up the society involved. In general terms what is needed is a system of taking the opinions and views of a group of individual people about an issue (typically which candidate to serve in some office) and obtaining a choice/candidate for the group as a whole. Of necessity, I will be compressing and oversimplifying the issues but a major goal is to show how mathematics can be used to guide and evaluate changes that might improve democracy here in America. The impetus for this essay is the recent step by New York City (NYC) in replacing its traditional system of using a "standard ballot" (vote for one person) and using ranked ballots instead. It is worth noting that there are many different rules that can apply to ranked ballots, sometimes also called ordinal ballots, as we will see below. In the new NYC system the votes are being counted by a system with various names but often called IRV for Instant Run-Off Voting. Here I will use the term sequential run-off to describe the method, but we will see that there are many variants of sequential run-off procedures. One goal is to show the pros and cons of different appealing voting methods. Burlington, Vermont also changed from plurality voting to sequential run-off (IRV) but an unexpected result led them to backtrack and return to plurality voting. While IRV is a better system than plurality voting in my view, there are better ranked ballot voting systems than IRV. I hope to convince you why with this account. Thinking about elections in a mathematical modeling environment will be our major goal but before one can conduct modeling, a certain amount of content-specific information is necessary. Issues related to electing a president of a club or a class at a school are rather different from elections for the legislature of a large country or the chief executive officer of a large democracy (President vs. Prime Minister). So I will begin with a brief look at different ways that people can be represented in building a democratic society.

Background

In a large democracy typically the populace elects a legislature (parliament) and there is either direct election for the chief executive (e.g. a state governor in the US) or there is a way that the legislature provides a way for a chief executive to be chosen. In democracies with ties to Great Britain there has been an emphasis on election where the legislative member represents some local geographic area. This is in contrast to the idea that if there were many parties with different shades of political views the members of the legislature for each party would be approximately proportional to the vote that party received in the election.

The chief executive in many European democracies is not elected in his/her own right but gets this position as a consequence of being the leader of a party or coalition of parties in the legislature. In America, in its early days political parties played a much less important role than they have in recent years. While people who were elected to the House of Representatives were affiliated with a party they were typically elected as a representative of a local district. In contrast in Scandinavia a person does not have a "local" representative in the Parliament. One votes for a party and if nationally there was approximately 30 percent vote for that party, then about 30 percent of the seats in the parliament would be assigned to that party, and specific people would be chosen from a list of names provided by the party. In recent years hybrid systems which try to build on the best features of these two different styles of representation in a legislature are being explored, for example in Germany.

Constructing a mathematical model for elections

What are the components that go into a voting system or a situation where some goal is accomplished by an election?

First of all one needs eligible people as voters. Dogs and whales display intelligence but they don't vote, nor do babies. However, there is a long history of groups of people who care about the issues involved in elections who have not been able to vote in certain elections or in particular periods of American history. Examples include women, Native Americans, the people who live in the District of Columbia and slaves. Here, however, since we will be looking at the election/voting process for those who can vote, we make a very simplifying assumption.

Voters

We assume that the number of people who can vote is a positive integer n which is more than 2. It is often convenient to assume that n is even to prevent ties in certain kinds of situations. We also note that for large elections the probability a tie will occur is low but it has happened, so some system for breaking ties should exist. For simplicity we will assume there are no ties or when there are, that we can break the tie at random, though the results of breaking ties in different ways may yield different outcomes.

!. There are n voters who can participate in an election, where n is a positive integer at least 2.

Choices (candidates)

The people who vote will be presented with a collection of choices. These choices might be candidates for the US Senate, what is the best play that opened on Broadway during the period Sept. 1, 2019 and March 20, 2020, or what kinds of food to serve at the the senior class picnic. Even when there are only two choices there are interesting questions to ask about elections but things get really interesting when the number of choices is 3 or more. Usually there is a system by which the items which could be voted for were selected. Thus, to run for the US Senate in a particular state would have requirements involving residency, age, etc. However, sometimes there have been provisions for the voters to amplify their choices by using a "write in" option.

2. There are m choices/candidates/alternatives to vote for, where m is a fixed positive integer at least 2, set prior to the running of the election.

Ballot

A ballot is a way for voters to express their opinions and feelings about the choices being voted on. Historically, there usually was only one type of ballot. The *standard ballot* asks that the voter choose one of the m choices to vote for. This ballot makes few demands on the voter but in an era where sometimes there are many choices to pick a winner from (for example, Democratic party primaries for the 2020 elections) the standard ballots give little information about the range of "views" a voter might have about the candidates/choices. In recent years there has been a huge explosion in suggestions for different kinds of ballots. These ballots vary in the amount of information the voter needs about the candidates to complete the ballot well and the ability the voter has to use this information about candidates to construct a ballot.

Two major types of ballots are ordinal (ranked) ballots and cardinal ballots.

Ordinal or ranked ballots

Figure 1 shows how one voter might cast an ordinal ballot involving three choices where A is preferred to C and B, and furthermore C is preferred to B. Note that in this ballot there are no ties among the choices, indicating that the voter might be indifferent between some of the candidates. The notation in Figure 1 is suggestive but takes a lot of room on the printed page, so another way of expressing the same information for this voter would be:

A > C > B

The > symbol, usually used to compare the size of two numbers, here is used to indicate that A is preferred to C and that C is preferred to B. Note that we will assume the preference is a transitive relation, that is, since A is preferred to C and B is preferred to C, A is also preferred to B. When experiments are done where a person ranks a large number of choices by considering preference between pairs, many times the ranking which results based on the pairs is NOT transitive.

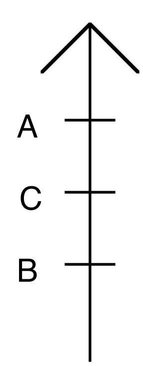


Figure 1 (Ordinal or ranked ballot with 3 choices; higher choices towards the

top.)

While it may seem that a voter can always produce such strict rankings that illustrate transitivity in the choices, an easy "experiment" allows one to see that things are not that simple!

Activity 1: **Paired Comparison Activity**

Suppose your feelings about fruit are being explored. Which fruits do you like more or less than other fruits? This is your "task." Given a pair of fruits, you have to decide which of the two fruits you prefer to the other.

Example:

Banana and Apple

If you prefer a banana to an apple, you write B > A and if you prefer an apple to a banana, you write A > B. You are not allowed to throw up your hands and say you like them equally. You are not allowed to say "I have never heard of these fruits and I cannot compare them." You cannot say, "I prefer a granny smith apple to a green grape but I prefer a red grape to a granny smith apple." The rules of this "game" are that you must decide which you like better for each pair, hopefully based on your true and honest feelings.

Here are the fruits to consider:

Apple
Banana
Cherry
Grape
Orange
Pear
Raspberry

Question 1:

To compare the fruits above in pairs, how many comparisons will you have to carry out?

Question 2:

Carry out the task of making all the required comparisons!

Question 3:

Based on the data you produced, which fruit is your most favorite?

Question 4:

Based on the data you produced, which fruit is your least favorite?

Question 5:

Use a directed graph (digraph) to represent the data you produced.

Question 6:

Is there a way to use the digraph you produced in Question 5 to determine your most favorite and least favorite fruits?

Question 7:

Based on the data you produced, can you produce a "ranking" of the seven fruits, that is, are you able to say which fruit you liked best, which you liked second best, ..., which fruit you liked least?

Question 8:

Give examples of "applied" situations where in essence the same task is being required but the ranking and paired comparisons involve something other than fruit.

Question 9:

The questions above are framed around the paired comparison of fruits. There are many other choices of things which might change your perceptions of paired comparison as an "effective" tool for obtaining preference information. For example, instead of fruits consider the following list of topics sometimes taught as topics in algebra/arithmetic/geometry to 8th grade students:

1 = solving linear equations
2 = fractions, ratio, and proportion
3 = slope of a line
4 = evaluate functions
5 = solving two linear equations with two unknowns
6 = Pythagorean theorem
7 = volume of prisms, pyramids and cones

Repeat what you did in Questions 1 - 8 where this time you interpret Topic A > Topic B (in the list above) to mean that you feel it is more important for 8th grade students to master Topic A than Topic B with no allowance for being indifferent and where you are working with topics rather than fruits. You have to decide what the word "master" means for you in this context.

It is not difficult to modify the notations we have used for ordinal preferences for allowing ties between the choices (candidates). Figure 2 shows an example. Here the voter is indifferent between A and C but prefers both to choice B.

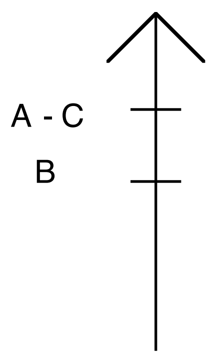


Figure 2 (Ordinal ballot involving 3 choices, where the voter is indifferent between A and C but prefers both of these to B.)

Another way that a voter could express this preference would be: A = C > B. Another sample of this notation is B > A = C. Note that by allowing indifference it is more likely that voters will be able to express a full range of reactions to the choices presented to them.

One further complication for using ordinal ballots is the feeling that voters be allowed to "truncate" their ballots, that is, not produce a ranking, with or without indifference, that lists all of the candidates. Why might a voter not want to rank all of the candidates? Think about the current US primaries (2020) for Democratic nominee for president, where there may in some primaries be as many as 20 candidates to rank. A voter might not know ANYTHING about some of the candidates and not list those unknown candidates on his/her ballot. The voter might, knowing the particular method that will be used to count the ballots, rank fewer rather than more candidates to avoid hurting the voters first place choice or second place choice of being the winner. So a voter, due to ignorance about candidate B in an election, might not want to produce the ballot shown in Figure 1 but rather use the truncated ballot shown in Figure 3.

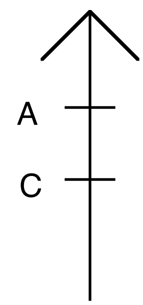


Figure 3 (An ordinal ballot where the voter has chosen to rank only two of the three choices. This is known as truncation.)

Or fearing that even this would give C some advantage regarding his/her first choice preference for A, would vote for A only (as in Figure 4).

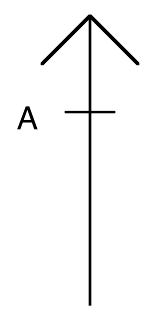


Figure 4 (This voter is using a ranked ballot but voted for only one of the three candidates available to choose from.)

It might be tempting seeing the ballot in Figure 4 to treat this as the same as the ballot:

A > B = C

but this to some extent changes the intent of the voter. On the other hand, some ways to count ballots, as we will see below, seem to have a different "feel" philosophically for truncated ballots than they do when ballots are "full."

For example, if a method involves looking at two-way races between candidates, then the ballot in Figure 1 would provide a count of 1 for A in the two-way race between A and C and a count of 1 for C in the two-way race between C and B. However, what does Figure 4 say about two-way races between and A and C and between B and C? Some might feel comfortable giving A a point (in methods that involve assigning point counts to candidates based on the ballots) in the race between A and C while others would not; for the race between B and C one could say they tied, but others would say that one can't say that. For point count methods some have suggested using fractions of a point for dealing with ties.

A typical instance of an election is the collection of ballots, of the kind specified by those running the election, where improper ballots have been removed from consideration. A typical election involving five candidates with full rankings by all of the voters is shown in Figure 5. Note that there are 55 voters but they only selected 6 of the possible 120 different choices a voter can have if there are no ties (indifference or truncation) when a voter casts his/her vote for 5 choices.

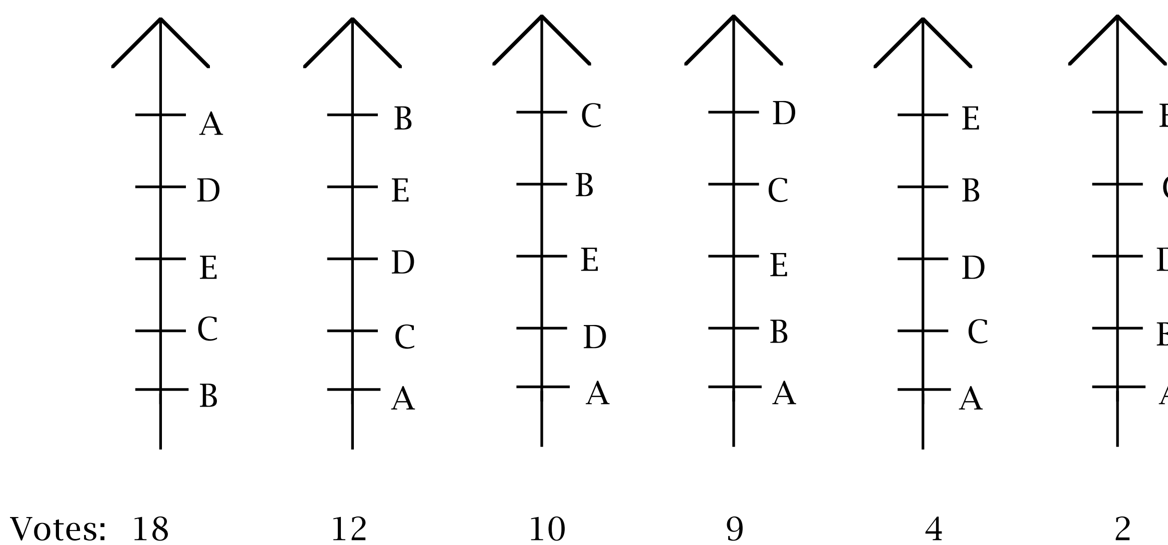


Figure 5 (An election where 55 voters ranked fully 5 different candidates. Six voters ranked E highest but they diverged about their lower ranked choices.)

Exercise:

(a) For 5 candidates, how many different ways can voters produce preference schedules where all of the candidates are ranked and there are no ties?

(b) Resolve part (a) where ties (indifference) are allowed between any group of candidates but all candidates are listed.

(c) How many different ways are there to vote for 5 candidates which consider full ranking ballots, indifference between two or more candidates, and truncations?

(Hint: You might want to solve this problem for 3 candidates before taking on the harder case of m = 4 or m = 5. You can also consider the case of finding formulas for any value of m which is 3 or more.)

You might want to give some thought to who you think is most deserving of winning the election in Figure 5 and what procedure will result in counting the votes that will result in this person's winning.

Cardinal ballots

Rather than provide a "preference ordering" for the candidates as in ordinal ballots, one might "grade" each candidate who is being voted on using some scale where it is easily understood and one can tell when one grade is better than another. For example here are three different ways to grade the three choices A, B, and C who are presented to the voters:

A gets a C grade, B gets a C grade and C gets a B grade.

Comment:

Note that two candidates can get the same grade, and that the scale here is A, B, C, D, and F (E omitted) and unlike grades for students, which sometimes allow + and - designations, here we did not allow that finer grading system. However, we could have used the scales

a. A+, A, A-. B+, B, B-, C+, C, C-, D+, D-, F

b. A, A-, B, B-. C, C-, D, D-, F

c. A, B+, B, C+, C, D,+, D, F

where the grades are better ones towards the left, and lower towards the right in a given string. Note the reason sometimes people argue for fewer rather than more items in the scale is that it is hard to tell for a particular candidate whether the candidate should get a grade of A- or B+. One also wonders whether for all people the "difference" between an A versus an A+ is the same as the "difference" between a B and B+. When grading a political candidate one may judge them on different issues, perhaps their views on tax policy, foreign policy and women's rights. You might see Candidate X as having A, A+, and C on these three issues and Candidate Y as having the grades B+, C-, and A+ and be faced with what grade to give under the allowed scale to the two candidates!

Other qualitative grading systems also exist:

d. highly disapproved, disapproved, neutral, approved, highly approved

e. very poor, poor, neutral, fair, good, excellent

In college it is typical to assign students letter grades for courses but in K-12 education it is often the case that a numerical grade is given for a student's work. Usually the grade assigned is some non-negative integer from 0 to 100. In this system the higher the number, the better the grade. However, once one gets the idea of using a numerical scale for grading there are in essence an unlimited number of choices, which raises the issue of the pros and cons of these different grading systems. Some examples of such systems are listed below.

f. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 with 0 the best grade

g. 10, 9 8, 7, 6, 5, 4, 3, 2, 1, 0 with 10 the best grade

h. 1, ..., 100 with 1 highest

i. 100, ..., 0 with 100 highest

j. -100, ..., 1, 0, 1, 2, ..., 100 with 100 the highest

k. -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5

So you get the idea. There are lots of different numerical scales which vary in the number grades that can be given out. But there are questions about how a voter reacts to different scales and if a voter will always have the skill to carry out the task of giving grades when there are many candidates running, and whether such grades are reliable. For example, if someone had to use the scale from 0 to 100 to rank the initial 20 candidates seeking the Democratic Party nomination for president in 2020 at 9 a.m., would the person give the same grades at 3 p.m. if not allowed to see his/her earlier grades, and give the same grades again at 9 p.m., without consulting his/her earlier grades. Also, will a voter look at some names they don't know and give that person a zero grade because they have never heard of the person or was the 0 a reflection of the fact that they knew the candidate and felt that this person deserved the lowest grade possible?

One grading method deserves special attention. It uses a ballot which is sometimes called an *approval ballot*. For each candidate the vote says that he is willing to have that candidate serve, that is approves of that candidate, or by not listing a particular candidate, the voter indicates disapproval (or ignorance about that candidate). This scheme, promoted by Steven Brams and Peter Fishburn under the name of approval voting, has garnered much attention for its simplicity in describing it. Although Brams and Fishburn may not have been the first to suggest something along these lines, they deserve great credit for calling attention to this approach and for trying to get it adopted as a system used in actual elections. Some professional societies now conduct some of their elections using Approval Voting, using an approval ballot. The winner is the person who gets the largest number of approval votes. Note that we can think of the approval ballot as a special kind of cardinal ballot where the scale has only two values.

l. 1, 0 (1 is better than 0)

It may seem at first glance that grading rather than ranking, using cardinal ballots rather than ordinal ballets might "dominate" using rankings but perhaps the case for this is not as easy to make as some claim.

Voter behavior

From the point of view of each individual voter he/she would like to see his/her most favored candidate triumph and win. Sometimes voters belong to some group whose interests the voter would like to see furthered. Men might prefer to vote for a male candidate even if they might see the male candidate running as not the "best," while women may support female candidates because they are female. Thus, voters, when asked to produce ballots, will sometimes vote in a way that does not reflect their individual honest choice but has a game theoretical or strategic purpose. One dramatic insight that mathematics has shown is that at the method level there are methods which, in order to decide a winner, involve more and more computation time as a function of the number of voters and/or candidates. Similarly, there are methods, which when fixed require more or less computational effort to decide what pattern of strategic (insincere for a purpose) voting will help a particular voter or collection of voters.

When an algorithm runs in polynomial time it is usually possible to solve large versions of the problem in a reasonable amount of time, while when an algorithm runs in exponential time, large instances sometimes can't be carried out. Usually, a problem instance has a parameter whose size affects the practicality of solving large instances. For elections the natural parameters are the number of voters and to a lesser extent the number of candidates. The class of polynomial time algorithms is often denoted P, and the class of algorithms which can be "checked" in polynomial time is call NP (non-deterministic polynomial time). NP is the collection of decision problems (answering a decision problem involves saying "yes," or "no") which have the property that when the answer is "yes," there is a proof this is the correct answer which can be verified in polynomial time.

A decision problem D is NP-complete if there is no known polynomial time algorithm to solve D nor is there a proof that an exponential time algorithm is required. Furthermore if any problem in this class could be shown to be solvable in polynomial time then this would be true of all of these problems. If any of these problems could be shown to require an exponential amount of work then all of the problems in the class would require an exponential amount of work. A major unsolved problem is whether the class P equals the class NP. A decision problem is NP-hard if, intuitively, it is at least as hard as any problem in NP. For an election method to be a practical one it can't belong to a class where it is computationally hard to find the winner.

Some voting methods are known to be NP-hard to decide a winner and some voting methods are known to be NP-hard to "manipulate." One wants to implement for actual elections decision procedures based on the electorate's ballots which can be carried out in a reasonable amount of time.

Goal of the election

After the m votes cast their ballots for the n candidates, the ballots must be used for a particular goal. This goal might be to elect a single winner, no ties allowed. This is certainly the most common type of voting we are familiar with but sometimes voting procedures use a ranking of the candidates. When voting is taking place for something like the best movie of the year, one might want to "output" from the results of the election a ranking of the all the candidates rather than a single winner. Sometimes the ballot counting might result in a collection of winners, indicating a tie for "best" choice. Or in some elections one might want to select exactly k winners (k at least 2) from among the m choices. This setting covers when one might be electing a committee of equals from the candidates to serve as representative at a college governance or in a union or corporation.

Unless otherwise mentioned here we will be concerned with selecting a single winner and often we will not specifically talk about what to do concerning ties but strictly speaking, any election method must have some way to break ties in case they do occur, even if this outcome (there being a tie) is extremely unlikely. Sometimes even in elections with large numbers of voters there may be a tie in "calculations" which are required to carry out this system. For example a system might use the number of last place votes for candidates as part of the system involved and there may be a tie among many candidates because they have the same number of last place votes.

What system should be used to decide the winner?

Our experience with elections usually leads to language of the kind, "let the 'best' choice win," or that the winner is the "voice of the people." We will explore various suggestions that have been made for how to decide elections based on ordinal ballots with no ties and no truncation. All of the systems described here can be modified to these other cases but it will simplify the discussion to restrict ourselves to the full rankings without ties case. (I will sometimes mention how to modify what is discussed below for these more general cases.) The case of deciding winners using cardinal ballots will also be briefly discussed.

Before discussing various methods let me remind you that what is being looked at here is modeling of elections rather than the way elections in the "real world" are carried out. For example, many of you have heard of run-off elections. In the real world what is often done is that if an election is held and no candidate gets a majority, a physical second election is held, often with the only names appearing being those two persons who got the highest number of first-place votes in the first round of the election. This approach is expensive, and rarely do very many people vote in the second election which calls into question the result of using this procedure. One of the practical aspects of having more information from the voters than can be gotten from the standard ballot is that procedures in the spirit of physical run-off elections can be carried out using an ordinal ballot which does not require that voters physically return to the polls. Ordinal ballots provide additional information so that if no candidate gets a majority, one can use the ordinal ballot to decide who the two people with the largest number of first-place votes were and use the ordinal ballots to decide the winner among these two choices (candidates). In passing, there are those who don't necessarily believe that when someone gets a majority of first-place votes this person should win. Such a person might get a slim majority and be hated by all the other voters while all voters might agree who the second best candidate would be and perhaps argue that this candidate is the better choice to win the election.

Election methods

In mathematical terms one can think of election methods, deciding a single winner based on the ballots (Figure 5 provides a sample instance of an election) as a *function* whose domain is the set of elections (an election instance being an instance of having each voter produce an appropriate ordinal ballot) with n voters where the range of the function is a single element of the set of candidates. So from this point of view we need to have methods that are "decisive," in the sense that given an input election there must be a single candidate output who wins the election. Again for simplicity we will often not discuss ties. One way to deal with ties, which is not altogether satisfactory, is to choose at random from among the tied elements. There are often advantages in using some randomness with regard to election methods but in "selling" such methods to the public it sometimes is hard to defend that the winner depends on "chance."

For teachers, elections offer a nonstandard way to introduce and discuss the properties of functions. One can think of an election method as a function whose domain consists of n ordinal ballots, one for each voter and whose range consists of a single element of the candidate set. If there are m candidates, each of whom produces an ordinal ballot without indifference or truncation involving any candidates, there are m! different choices for each voter, and since there are n voters there are $(m!)^n$ different elections as inputs to the function. Thus, there are an amazingly large number of different election methods possible, mostly ones that will not occur in a real election.

What follows is a collection of methods that have been suggested, with those at the beginning being ones that have a longer history or have been actually used in the "real world" at some time in the past. The descriptions I use apply to the situation where the voters use ordinal ballots where all of the candidates are ranked. The examples I use don't allow indifference on the part of the voters but the descriptions can usually be easily modified to allow indifference between candidates. Some of these methods seem "philosophically" different in spirit when truncated ballots are allowed. After a discussion of the mechanics of conducting different elections and with brief comments about their "appeal," I will discuss the different properties of the decision methods which have been offered for why they should be adopted. I will also discuss the contributions of Kenneth Arrow (1921-2017), Alan Gibbard and Mark Satterthwaite who proved theorems which show the limitations of these methods of deciding elections in having all of the appealing properties that one might like them to have.

For what follows, if you want to check your understanding of the methods you can refer to the election in Figure 5 (five choices) or Figure 6 (three choices).

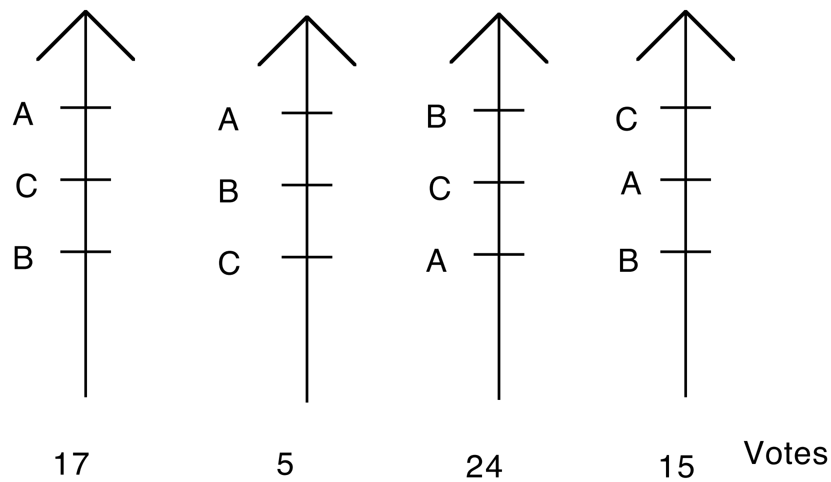


Figure 6 (An election where 61 voters have cast 4 different types of ballots among the 5 types they might have produced.)

Remember, we will assume there are no ties and that each voter produces a full ranking of all of the candidates (no truncation) using an ordinal ballot.

1. Plurality voting

The winner is the candidate with the largest number of first-place votes.

Comment:

Nearly all elections in America, which typically don't use ordinal ballots are decided by plurality voting. However, when there are ranked ballots the method still works. Note that when there are many candidates some candidate can win with a very small percentage of the first-place votes. (For 10 candidates, if the vote splits quite evenly, someone might be the plurality winner with only 11% of the vote. And the plurality winner might not be very popular with all of the other voters (low ranking on each ballot).

2. (Standard) Run-off election

If no candidate gets a majority of first-place votes one can have a "run-off" between the two candidates with the largest number of first-place votes. (The winner of this two candidate election, where someone must have a majority when the number of voters is even, is the winner.)

Comment:

In the real world a run-off election often requires that the voters come to the polls again. Running a second election is very expensive and typically attracts many fewer (and perhaps different) voters from the first election. However, this "inefficiency" can be avoided if ranked ballots are used. One can eliminate the candidate names from the ballots except the two highest vote getters and determine who wins in the 2-way race between these two candidates. We will see later that run-off elections conducted in this way have some aspects which are not ideal.

After noticing the concept of a run-off election one can perhaps be bothered by the fact that all candidates except the top-two vote getters are eliminated at once. This can be handled if one wants by using a run-off system that eliminates candidates one at a time. There are different names for this system, most commonly IRV for "Instantaneous run-off voting" but I like the more suggestive name Sequential run-off voting.

3. Sequential run-off (more commonly called IRV)

If no candidate has a majority of first-place votes then one selects the candidate with the fewest first-place votes at this stage and eliminates this candidate. The procedure is repeated until a single winner emerges. (At each stage, after a candidate is eliminated, check if some candidate now has a majority or not.)

4. Condorcet voting (Named in honor of Nicholas de Coridat, Marquis de Condorcet (1743-1794) who encouraged the use of this method.)

The winner is the candidate, if there is one, who can beat each other candidate in a two-way race.

Comment:

Perhaps unintuitively it is not always the case that there is such a candidate. For the election in Figure 7 we have that A beats B in a two-way race by 39 to 24; B beats C in a two-way race by 42 to 21; C beats A in a two-way race by 45 to 18. When A earns more in a year than B, B earns more in year than C, then A earns more in a year then C. In general, if one has a relation R, where a R b and b R c implies a R c we say that R is a transitive relation. Many of the relationships we are most familiar with in the world and to some extent in mathematics are transitive. For example, equality of numbers and parallelism of lines in the Euclidean plane are transitive relationships. However, two-way race results for ordinal ballots sometimes obeys transitivity and sometimes it does not.

The election in Figure 7 offers an example of an election with no Condorcet winner when there are 3-candidates.

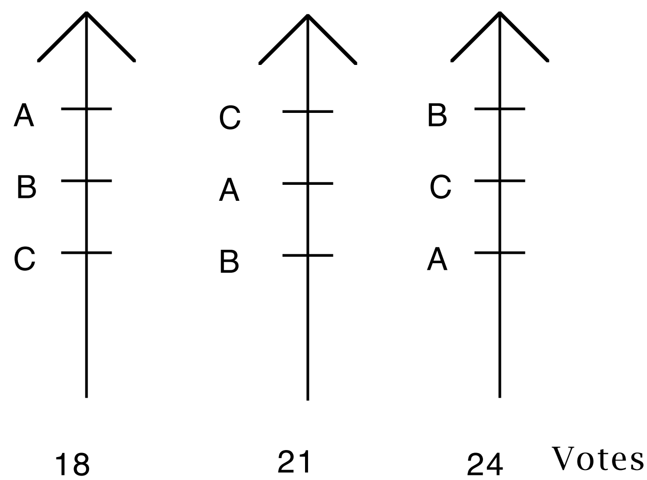


Figure 7 (An election where no candidate can beat all of the others in two-way races.)

As mentioned above an easy computation shows that A beats B in a 2-way race by 39 to 24; B beats C in a 2-way race by 42 to 21; C beats A in a 2-way race by 45 to 18. We can display the results of the two-way races using a directed graph (digraph), a diagram with dots and line segments with arrows. This digraph is shown in Figure 8. The fact that there is no Condorcet winner shows up in the diagram with the fact that there is a directed cycle (some call it a directed circuit) of length 3, as shown in Figure 8. Graph theorists call digraphs of this kind complete directed graphs or tournament graphs. When there are n vertices the graph can show that there is a ranking where there was a winner, a single second-place choice, down to a candidate (team" that lost all of its games. Even if there is a candidate who won all two-way races as shown in the pairwise vote digraph, there may be cycles or circuits among the other candidates.

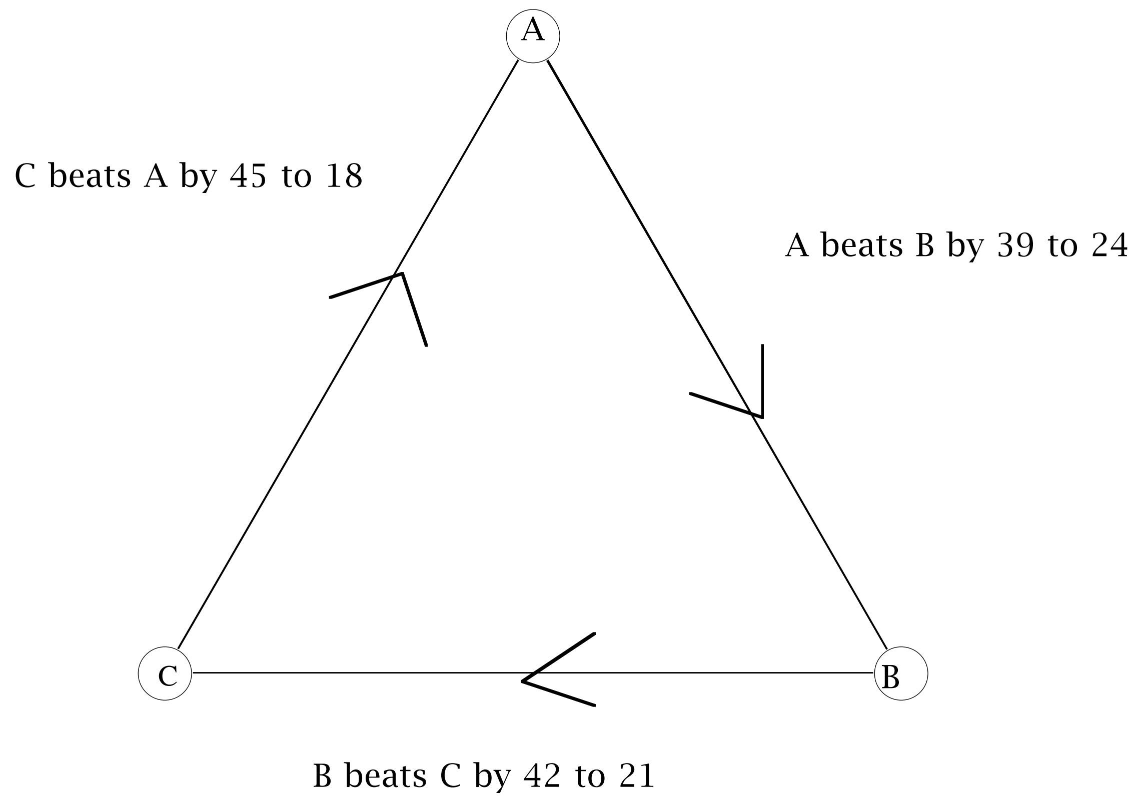


Figure 8 (Pairwise race digraph showing that there is no Condorcet winner for the election in Figure 7.)

Another way to display the results of the two-way races which will come in handy is to use a pair-wise preference table (matrix). The data for the election in Figure 7 is shown in the table below (Figure 9).

	A	B	C
A	-	39	18
B	24	-	42
C	45	21	-

Figure 9 (Pairwise preference matrix. A is preferred to B by 39 voters and B is preferred to A by 24 voters.)

The entry in the ith row and the jth column is the number of voters who preferred candidate i to candidate j. Thus, in Figure 9 the entry in row A and column B is 39 while the entry in row B and column A is 24. Note that the sum of these numbers is the total number of persons who voted in this election instance. Note that if one knows the total number of voters and the entries either above or below the diagonal of the table, one can recover the other entries.

It is not difficult to extend the idea used to construct the election in Figure 7 to elections with more candidates. Here is the way that the preferences in Figure 7 were generated. Starting with the schedule on the far left, take the bottom choice and move it to the top without altering the relative positions of the other candidates. Thus, we started with A > B > C, and we would obtain the second column of the Figure 7, the preference schedule C > A > B, and finally, moving B to the top and pushing down the other choices we get the preference schedule B > C > A. By assigning the first schedule with some number of votes and the other schedules different but approximately equal numbers of votes (best to use numbers which sum to an odd number of total votes), there will be no Condorcet winner.

For example, to get a four-candidate example, starting with A > B > C > D (say with 11 votes) we get the schedules D > A > B > C (say with 12 votes), C > D > A > B (with 13 votes) and B > C > D > A (with 19 votes); you can verify that the resulting election has no Condorcet winner.

Note that if there is no Condorcet winner a way to pick a winner must be found because it is important that for any election (collection of ballots submitted by the voters) a winner be chosen using a method that the voters have been informed of in advance. After finishing this introduction of appealing methods, I will return to this issue.

The next method is based on the idea that some candidates are higher on voter ballots than others. One could try to choose a winner based on using this information as part of the voting system.

5. Borda Count (Name in honor of Jean-Charles de Borda (1733-1799) who encouraged its use.) (Described for ranked ballots with ties.) (Some provision has to be made for when there is a tie in the total Borda Counts for two candidates.)

For each voter's (fully ranked or allowing ties) ranked ballot, count points for each candidate as the number of candidates on the ballot who is below the given candidate. The point count for a candidate is the sum of these numbers over all voter ballots. The candidate with the highest number of points wins.

Comments

Example: For the ballot in Figure 1, A is given 2 points, C is given 1 point and

B is given 0 points. For the ballot in Figure 2, A is given 1 point, C is given 1 point and B is given 0 points. For the election in Figure 7 the computation of the Borda Count for each of the three candidates is as follows:

A = 2(18) + 1(21) + 0(24) = 57 points

B = 1(18) + 0(21) + 2(24) = 66 points

C = 0(18) + 2(21) + 1(24) = 66 points

In this election based on the Borda Count there would be a "tie." This occurs despite the fact that there is no tie for first-place votes. One approach to breaking a tie is to use first-place votes to help break a tie. In this case since B has more first-place votes, B would be the winner, but the Borda Count might give a tie, and there might also be a tie for first-place votes.

One can take the Borda Count and divide it by the number of candidates involved, giving a number which represents an "average" rank for the candidates, and electing the candidate with the highest average rank on voter ballets. Working with the initial Borda Count or average will always pick the same candidate, since the division does not affect which candidate achieves the maximum.

Another interesting fact here is that for this way of computing the Borda Count when all candidates are ranked equals the sum of the entries in each row of the pairwise preference matrix (see Figure 9 for the example just discussed). We get each candidate's Borda Count from the row sum. This is a general theorem that can be proved mathematically.

Many descriptions of the Borda Count (when all candidates are ranked without indifference) assign points on the basis of giving a particular number of points for a first place, a smaller number for a second place, and so on. There are interesting mathematical questions about whether different ways of assigning points yield the same winner or ranking based on the Borda Count.

Should one provide honest rankings when one knows the Borda Count is to be used? If one has poll data indicating that the person Z you view as being 3rd best is doing very well, does it make sense for you to rank Z last in the hope this will help your first or second choices win? Yes, this might make sense but other voters may adjust their votes strategically resulting in an outcome that is far from what represents the "true" preferences of the electorate. This strategy, putting candidates relatively high in your ranking lower, in the hope it will help your favorite candidate is known as "burial."

Activity 2: **Election Methods**

1. Who do you think should be the winner of the election shown in Figure 5 based on the preferences that the 55 voters have expressed? What is the reasoning you used for selecting who you think is more deserving of being the winner?

2. Use the plurality method, run-off, sequential run-off, Condorcet, and Borda Count to determine a winner for the election in Figure 5.

Note: If you have done the computations correctly, the five different election methods will produce 5 different winners!

3. Discuss the implications of the reality that for the same set of ballots different election methods choose different winners. In light of this, how should we choose which election method to use?

More election methods

We have looked at five of the most common and appealing methods of deciding the winner of an election based on full rankings using ordinal ballots. Briefly, here is an additional sample of methods which have been suggested or explored over the years. Some of these methods have been developed because they select a Condorcet winner when there is one, a feature which some theorists find appealing but not everyone thinks it important to elect a Condorcet winner when there is one.

6. Fewest last-place votes (anti-plurality)

The winner is the person with the fewest last-place votes.

Comment:

This often gives rise to many ties. There may be many candidates who get no last-place votes.

7. Coombs method (Named for Clyde Combs (1912-1988))

Repeat: If a candidate has a majority this candidate wins. If no such candidate exists, eliminate the candidate with the *largest* number of last-place votes.

Comment:

Coombs need not elect a Condorcet winner when there is one.

7. Borda Count Run-off (Baldwin's Method. Named for Joseph Baldwin (1878-1943), an Australian.)

That candidate who wins is the one who emerges from a series of run-offs based on the Borda Count: at each stage the candidate with the lowest Borda Count is eliminated, and new Borda Count computed based on there being one less candidate, the eliminated candidate being removed from the ballots without changing the relative positions of the remaining candidates.

Comment:

One can prove that if there is a Condorcet winner, this candidate will emerge as the winner. However, when there is no Condorcet winner, someone will win.

8. Bucklin (Named for the American James W. Bucklin (1856-1919).

If someone has a majority of the first-place votes, that person is elected. If not, consider first and second-place votes. If someone has a majority, elect that person (if more than one such person exists, elect the person with the largest majority), if not, continue to consider first, second, third-place votes until a winner emerges.

9. IRV-bottom

If someone has a majority elect that person. If not, hold a run-off between the two lowest first-place vote getters. Eliminate the loser. Repeat.

10. Benham-IRV (Invented by Chris J. Benham)

If there is a candidate who can beat everyone else in a two-way race, elect that person; if not:

a. Choice X beats choice Y means that the number of ballots ranking X over Y is greater than the number of ballots ranking Y over X.

b. Repeat sequential-off (IRV) (based on fewest first-place votes) until there is one un-deleted candidate who beats each one of the other un-deleted candidates. Elect the candidate that remains.

11. Black's Method (Named for Duncan Black (1908-1991), a Scottish economist.)

If there is a candidate who can beat all of the others in a two-way race, elect that person. If not, use the Borda Count to obtain a winner.

Although not everyone thinks electing a Condorcet winner when there is one is important, I think this is an appealing "rule" to require. Various methods have been proposed which, while not part of their description, guarantee a Condorcet winner will be elected when there is one. We have already seen one example of this, what is called Baldwin's Method above — a run-off based on the Borda Count. Here is another example of such a method.

12. Nanson's Method (this name has been given to several methods which are not the same); here the case where all the candidates are ranked and no indifference is allowed is the one looked at. Named for Edward J. Nanson (1850-1936), an Australian.)

(a) Compute the mean value M of the Borda Count for the candidates, that is, add the Borda Count for each candidate and divide by the number of candidates.

(b) Eliminate from the ballots the candidates which are less than or equal to M. Compute the Borda Count of the remaining candidates.

(c) Repeat (a) and (b) until a single candidate is left.

Comment:

It can be shown that if there is a Condorcet winner, this candidate must win in the Nanson procedure.

15. Dodgson's Method (Name for C. L. Dodgson (1832-1898), a British mathematician better known to the public as Lewis Carroll, the author of Alice in Wonderland.)

If the election instance has a Condorcet winner, that candidate is selected. If not, the winner of the election is that candidate for which a minimum number of swaps makes that candidate a Condorcet winner.

The idea of a swap is that for one voter, two adjacent candidates in the voter's ordinal ranking are switched.

Given an integer c and an election instance it is NP-complete to determine if a candidate can become a Condorcet winner with fewer than c swaps. Thus, some elections may take a long time to decide using this method.

14. Kemeny-Young (John Kemeny (1926-1992), Hungarian-born mathematician and computer scientist. H.P. Young is an American (now living in England, combinatorist, economist, game theorist)

Start with defining the "distance" between two voter ballots (all candidates ranked; no ties). This is usually done by: if two ballots differ on the ranking of x and y, then one assigns this pair a 1; if the ballots agree, this pair gets a 0. Sum the values for all pairs of candidates to get the distance between the two ballots. The winner of the election is that ranking which minimizes the total distance between the proposed winner and all of the ballots cast.

Comment:

A winner is computationally hard to compute for this method, but it does have the appealing feature that if there is a Condorcet winner, this candidate will be elected.

When considering different methods which elect a Condorcet winner when there is one but which are decisive and will choose some other winner when there is no Condorcet winner, one might pick a method which discourages strategic voting. While it turns out that all voting methods which are not "dictatorial" are subject to manipulation using strategic voting, some of these methods are much more immune to strategic voting than others. For example, it may require more computational resources to figure out how to optimally vote strategically with one method than with another.

Being fair and democratic

The surprise from the analysis so far is that different seemingly reasonable and appealing methods of deciding an election on the basis of a collection of ballots (an election) can yield different winners. This being the case, how should we judge when one election decision method is better, fairer or more democratic than another method.

A systematic way of addressing this question was carried out by Kenneth Arrow, a mathematical economist. Arrow studied mathematics as an undergraduate at City College of New York, now part of the City University of New York and continued his student career by getting a doctorate in economics at Columbia University. His thesis adviser was Harold Hoteling a mathematician and statistician. His thesis dealt with the issue of making group decisions on the basis of the views/opinions/ballots of a group of individuals. What did Arrow do?

Arrow considered the situation where a collection of individual voters produced ordinal ballots with ties (indifference between choices) allowed. He then suggested axioms or rules that a decision method used to rank the choices on the basis of the input of the individuals should obey. HIs work was concerned not with finding a "winner" among the different choices but a ranking of the choices, again, where ties were allowed. Rather than use the axioms that he employed I will give a variety of examples of the kinds of rules that Arrow and latter contributors to the theory of voting/elections have championed. I will also state axioms in the environment where the individual ordinal ballots are used to select a single winner from among the candidates or choices. Following Arrow's pioneering work there was another thread of interest in election and voting that dealt with the advantages or disadvantages of not voting in a sincere or honest way. This is often called strategic voting and it suggests that voters might try to achieve some better outcome by not being truthful about their true feelings about the candidates. This work is associated with the names of the philosopher Alan Gibbard and the economist Mark Satterthwaite.

SPOILER ALERT

The results below occur for an election where there are three or more candidates/choices. For Arrow's result attention is restrict to ordinal ballots but the Gibbard/Satterthwaite result holds more generally:

(Arrow) There is ***NO*** voting method (using ordinal ballots) when a decision involving at least 3 choice/candidates is required which obeys a short list of fairness/consistency/democratic requirements!

(Gibbard-Satterthwaite) This is ***NO*** voting method (using ordinal or cardinal ballots) when a decision involving at least 3 choice/candidates is required where voters can't improve their outcome by using strategic voting (producing ballots which differ from true information about the candidates) other than DICTATORSHIP!

Sometimes Arrow's Theorem is described by saying that when there are 3 or more choices there is no perfect voting method. Some people seem to think that this means that because there is no perfect method, plurality voting should be used. However, nearly all scholars agree that plurality is a particularly bad choice. The problem is that there is no universal (or wide) agreement about what method that improves over plurality should replace it. In light of Arrow/Gibbard/Satterthwaite one can't have everything one would like. So no consensus has emerged as to how to move forward to a better system. A major issue with the Gibbard-Satterthwaite result is that to vote in a strategic way for one's advantage one needs information about how other voters will vote, and one can't be sure if they will vote honestly or strategically. Where does one get such information about other voters' behavior? One must either use "reliable/accurate" polls or deduce information from "circumstantial" evidence. Thus, one might reason that "conservative" voters given this set of choices will do X, while "progressive" voters given this same set of choices would do Y, where X and Y cast their ballots on the choices in a particular way.

Thought provoking examples

Before spelling out the "abstract" fairness/consistency rules that form the basis for the theoretical work of Arrow/Gibbard/Satterthwaite, first I will consider an extensive collection of elections which, when decided by the appealing methods we have looked at, we get results which are unintuitive or make us nervous that the method is not really as good as we thought.

Example 1

The Election in Figure 5

Unexpected behavior:

The 5 appealing methods, plurality, run-off, sequential run-off, Borda and Condorcet each results in a different winner! The voter ballots are important but so is the method that is used to tally the votes.

Activity:

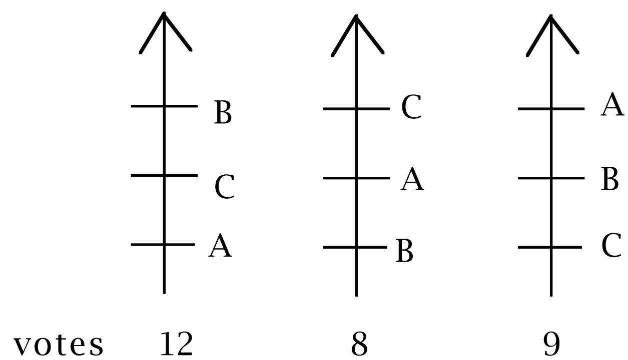
Determine the winner of the election in Figure 5 using;

a. Coombs b. Bucklin c. Fewest last-place votes d. Baldwin

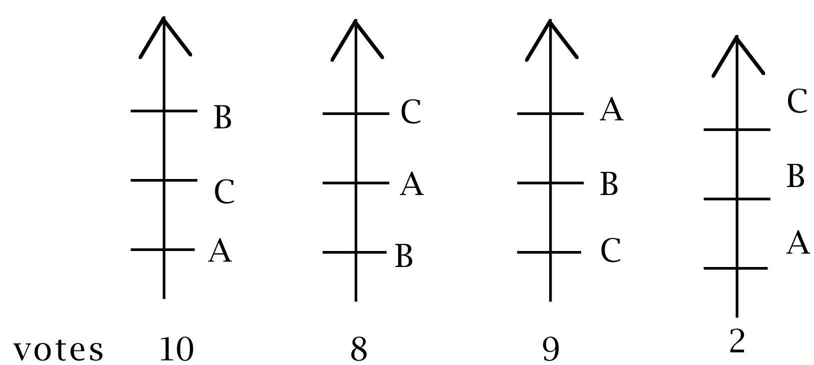
Example 2

The following example designed by Warren Smith raises some important questions for supporters of some proposed "reforms" to plurality voting.

The way that the two elections in Figure 9 differ is that in the second two of the voters who originally voted B > > C > A now vote for the ballot C > B > A instead. One can state that in going from the first election to the second B, has lost some support and A has gained some support. Monotonicity is the notion that the election method used won't hurt a candidate who gets more support or help a candidate who gets less support



(a)



(b)

Figure 9 (A pair of elections which can be used to illustrate the notion of monotonicity of an election method.)

Activity

(a) Just looking at the election in Figure 9(a) who do you think should be the winner?

(b) Just looking at the election in Figure 9(b) who do you think should be the winner?

(b) Compute the winner of the election in Figure 9(a) using sequential run--off (since there are only three candidates this coincides with using run-off) How does the winner you find compare with your view about who you expected to win?

(c) Compute the winner of the election in Figure 9(b) using sequential run-off. How does the winner you find compare with your view as to who you expected to win?

(d) Is there some unexpected behavior here? Can you articulate what the problem is?

Example 3 (Chris Benham)

Honest ballots:

46 votes A>B>C
44 votes B>A>C
5 votes C>A>B
5 votes C>B>A

Insincere strategic ballots:

46 votes A>B>C
44 votes B>C>A
5 votes C>A>B
5 votes C>B>A

i. Who do you think "deserves" to win each of these elections?

ii. Compute the winner of these two elections using:

a. Plurality b. Run-off c. Sequential run-off (IRV) d. Borda e. Condorcet f. Black, g. Baldwin h. Coombs i. Bucklin

Example 4 (Chris Benham)

Honest ballots:

49 votes C>A>B
31 votes A>C>B
17 votes B>A>C

3 votes A>B>C

Insincere strategic ballots:

49 votes C>B>A
31 votes A>C>B
17 votes B>A>C
3 votes A>B>C

Example 5 (Warren Smith)

Election 1:

9 votes for A>B>C
12 votes for B>C>A
8 votes for C>A>B

Election 2

14 votes for A>B>C
7 votes for B>C>A
8 votes for C>A>B

Example 6 (Warren Smith)

Suppose a certain town has 3 boroughs and the voting is recorded for borough:

West:

3 votes for A>B>C
4 votes for B>C>A
4 votes for C>A>B

South:

2 votes for A>B>C
3 votes for B>C>A
4 votes for C>A>B

East:

4 votes for A>B>C

5 votes for B>C>A

i.

a. Who do you think should win in each borough?
b.. Use sequential run-off (IRV) to decide the winner in each borough.
c. If you lump together the votes for all three boroughs, who do you think should win city wide?
d. If you lump together the votes for all three boroughs, who wins when you use IRV?

ii. Again, who do you think "deserves" to win each of the four elections (each borough, and lumped information) in Example 6?

ii. Compute the winner of these 4 elections (each borough, and lumped information) using:

a. Plurality b. Run-off c. Sequential run-off (IRV) d. Borda e. Condorcet f. Black g. Baldwin h. Coombs i. Bucklin

Example 7 (Forest Simmons)

i.

Who do you think should win this election?

45 A>B>C
20 B>C>A
35 C>A>B

ii. Determine the winner of this election using:

a. Plurality b. Run-off c. Sequential run-off (IRV) d. Borda e. Condorcet f. Black, g. Baldwin h. Coombs i. Bucklin

iii. Suppose that A were to drop out of the race. Who would the modified election for each of the methods above select as a winner. Compare the original result with the new result.

Fairness axioms for voting

What follows is more detail about the rules/axioms one would like election methods to obey, but the statement of these rules depends on the exact setting one is looking at — ordinal vs. cardinal ballots, truncation allowed or not allowed. I will more or less confine my description to the case where voters produce full rankings with no ties or indifference and ordinal ballots are used — elections like the ones in Figures 5 and 6.

Decisiveness and all legal ballots counted

The election decision method given any set of ballots should result in a single winner, and any ballot that obeys the ballot rules must be counted.

Comments:

While ties can and do occur, one can argue that when the number of ballots is large, the chance of such ties happening is low. The goal here is not to give the "board of elections" the opportunity to change the method on the fly or to argue that some ballots make no sense (30 candidates with a ranking alternating extreme progressives with extreme conservative candidates) and thus will not be counted. In mathematical terms, requiring that the election decision method be a function means that this requirement will be obeyed. However, the Condorcet method is not decisive unless there is some provision of what to do when there is no Condorcet winner. Below I will discuss a variety of modifications of "vanilla Condorcet" which elects a Condorcet winner when there is one but chooses some candidate as a winner in all elections.

Winner is not imposed

Comments:

In ancient Greek literature there is discussion of when someone had an important decision to make, the person would go to the Delphi oracle for advice about what to do. Making decisions is hard and seeking help from those with "wisdom" is sometimes a good idea. However, with regard to elections the winner must be determined using a system that involves using the ballots. Going to an "oracle" is not allowed.

One form of this idea of not being imposed is that if all the voters list candidate X at the top of their ballot, then X should win. To violate this condition would mean in essence that the ballots cast by the voters is not being taken seriously!

More support should not hurt a candidate (monotonicity)

Comments:

Suppose there are two elections E_1 and E_2 where in the first (E_1) the election method results in candidate C winning. The only difference between the two sets of ballots is that in E_2 C is at least as high on each ballot as in E and the relative positions of all the other candidates is unchanged (see Figures 10 and 6 for an example). Intuitively, more support should not harm candidate C. Now for the method used to be "monotonic" the winner of the E_2 election should be the same as when the method being examined is applied to E_1. Note that some methods may, when ties occur, involve the use of chance and we don't want an anomaly to occur because chance breaks ties differently, so we are interested in the situations where chance did not decide the winner.

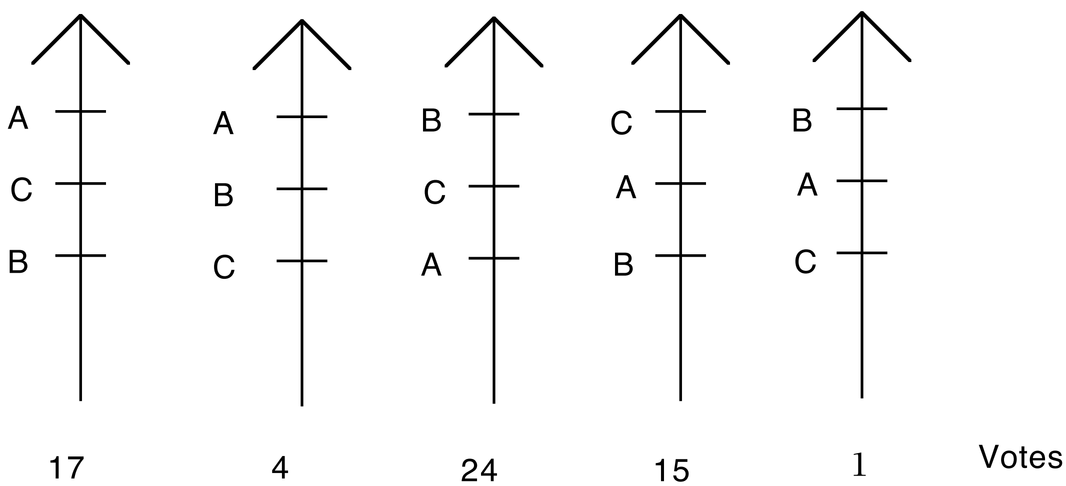


Figure 10 (A modified version of the election in Figure 6, designed to show that some voting methods don't obey monotoncity.)

The change which occurs is that B is now higher on one ballot than before. A "good" method would not "penalize" B for doing better. This rule does not say who should win in a given election that changes in ballots be treated in a "consistent" way. Similarly, if the only difference is that a candidate has been moved down in some ballots, that should not change this candidate from a loser to a winner.

Exercise:

i. Who do you feel should win the elections in Figure 6 and Figure 10?

ii. Compare the winners for the elections in Figure 6 and Figure 10 when the following methods are employed:

a. Plurality b. Run-off c. Sequential run-off (IRV) d. Borda e. Condorcet f. Black g. Baldwin h. Coombs i. Bucklin

Independence of Irrelevant alternatives

The relative position in a ranking for two candidates X and Y, based on using the given election system, does not depend on the presence of some additional alternative Z.

Comment:

This is not quite the wording Arrow originally used, and remember that here we are concerned with a single winner, but Arrow was concerned with a ranking for "society" based on the votes of the individuals who made up the society of group. The initial work of Arrow had some "minor" mistakes in it which were corrected. Over the years various new proofs and slightly different settings for his work have been described. Arrow won the Nobel Memorial Prize in Economics in part for his work related to elections and voting.

This rule (axiom) is probably the most "controversial" of those Arrow suggested.

Consider the following two elections, each of which has three voters who rank 3 candidates, A, B and C.

Election 1:

Voter 1: A > B > C
Voter 2: B > C > A
Voter 3 C > A > B

Election 2:

Voter 1: A > B > C
Voter 2: B > A > C
Voter 3 C > A > B

Note that with regard to the candidates B and C in both elections Voters 1 and 2 prefer B to C while Voter 3 prefers C to B in both.

Now, let us compute the Borda Count for the two elections:

Election 1

Borda Count

A = 2+0+1 = 3
B = 1+2+0 = 3
C = 0+1+2 = 3

Thus, there is a three-way tie. In particular for B and C neither would be "preferred" to the other based on the Borda Count.

Election 2

Borda Count

A = 2+1+1 = 4
B = 1+2+0 = 3
C = 0+0+2 = 2

Thus, for election 2 the "group" prefers A to B to C!

The only difference between the two elections is what one voter thinks about alternative A compared with C. Thus, for B and C, A is NOT irrelevant, and the Borda Count violates this "fairness" or "consistency" rule.

Suppose you go to a restaurant and they offer chicken (C), beef (B) or ham (H) sandwiches.

You tell your friends, given these choices, I would rank the three choices:

C > B > H and, thus, tell the waiter that you would like a chicken sandwich.

The waiter returns and states that unfortunately, the chicken option no longer exists; what would you like to order?

You answer, please get me a ham sandwich. Do your friends have a right to be confused? For the choice between B and H, C seems an "irrelevant" alternative.

Some discussions of the Independence of Irrelevant alternatives also raise issues concerning the idea of interpersonal comparison of utilities. How much satisfaction or "utility" might you get from eating a prune versus an apricot of equal weight? Perhaps you would say the prune is worth 7 points to you while the apricot would be worth 4. However, your identical twin sister might answer 6 points for the prune and 5 points for the apricot. Does this mean that you like prunes more than your sister likes prunes? Some people would say yes, and some people would say, not necessarily.

What Arrow showed was that the axiom of Independence of Irrelevant Alternatives was related to the questions of whether one could or could not make meaningful distinctions between the utilities that people assign to things. A similar issue comes up in judging "pain." If two people (identical twins) are subjected to the same painful experience (perhaps an electric shock of V volts for 3 seconds). You say the pain is a 7 and your twin says it was 5 on a scale from 10 to 0, 10 highest. Not everyone will think this means that you had a more painful experience than your twin sister.

Majority

If a candidate get a majority of the first-place votes that are cast, this person should win the election.

Comment:

The Borda Count does not obey this condition.

Over time more and more desirable properties of election methods have been identified. Typically some elections obey these properties and some don't. It is fun to try to find examples of elections that don't obey particular properties or groups of properties. And sometimes one can prove theorems about the interplay of the properties and different methods. One can think of Arrow's Theorem as being in this category.

Condorcet winner

If there is a Condorcet winner (someone who beat all other candidates in a two-way race) the method should elect that person.

Comment

Plurality, sequential run-off (IRV), Run-off, and Borda do not obey this condition. Baldwin and Nanson's methods do obey this condition.

Condorcet loser

A choice (candidate) is a Condorcet loser if that choice loses in every two-way race. A method which can elect a Condorcet loser seems not to have much to say for it.

There are many more fairness rules one might like to see an election method obey. However, the names and the details of the nature of these axioms is not totally standardized, partly because some of the axioms are stated in the case where rankings of all choices are given without indifference (ties), some are started when indifference is allowed, and others also allow truncation of ballots. You can find more examples in the references.

Elections are important in a democratic society. The fact that America uses plurality rather than a method which would do a better job in reflecting the "mood" of the country as expressed in using ordinal ballots, might have made for better times for larger groups of Americans.

Elections using cardinal ballots.

There are a large many cardinal ballot methods that have been described as ways to improve over the standard ballot and ordinal ballots. Let me say a few words about some of these.

Score or range voting (Using the scale 0-9, but using 0-99 is also a common suggestion.) It is promoted by Warren Smith.

Each voter grades each choice (candidate) with a number from 0-9. The points for each candidate are summed and the person with the highest mean (average) score wins.

Comment:

Warren Smith has studied this system extensively and gives arguments for its superiority over ordinal methods and other cardinal methods. This method is subject to strategic voting and requires that voters be able to decide whether to give candidate Z an 11 or 12 when they cast their ballot.

Approval voting (Steven Brams and Peter Fishburn)

Each voter decides whether to approve or not approve of each candidate. The candidates voted for by a person can be interpreted as those candidates the voter is willing to have serve in the office being filled.

Comment:

Approval voting is similarly easy to explain to voters as plurality. It is subject to strategic voting.

Majority Judgment (Michel Balinski (1933 -2019) and Rida Laraki)

Using a scale from 1 to 6, where 6 is best, grade each candidate.

For a collection of numbers X, sorted from smallest to lowest, if there is an odd number of numbers, the middle one is the median; if there is an even number of elements in X there are two numbers "in the middle." When the median is used below, take the lower of these two middle numbers as the median choice when the size of the collection of numbers is even.

For each candidate, compute the median value of the grades he/she is given. The winner is the candidate with the largest median score.

Note: This method often results in ties. Various ways to break ties, have been proposed, and the method of Balinski and Laraki is somewhat complex and not given here.

Voting and elections are critical for America's continuing functioning as a democracy. As America has become more polarized politically, having election methods which are responsive to the range of voter preferences seems worthwhile. After all, in two recent Presidential elections candidates who got the largest number of popular votes did not win (e.g. Albert Gore and Hillary Clinton). I have tried to survey here some of the many issues that a mathematical analysis into elections and voting have raised. In fact, there is much more to be said. There are lots of materials on line, and the references below give a small sample of the sources that are available. There are lots of fascinating ideas to explore both from a mathematical point of view and from a fairness point of view.

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