Pairwise Equity in Apportionment (2020)

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Suppose some apportionment algorithm (which assigns claimants positive integer amounts of some "prize" of positive integer size h) has been used to assign a collection of seats to parties in an apportionment problem situation.

Suppose A gets a seats and B gets b seats (where a and b are positive integers).

B claims to a court that A got more than it was fair to give it. B claims a better distribution of seats would be A gets a-1 and B gets b + 1 seats. How might the court evaluate B's claim?

Let us consider a specific example. Suppose A, B, and C have votes of 1200, 900, and 303, respectively.

Divide by:	A	B	C
	1200	900	303
1	1200 1	900 2	303
2	600 3	450 4	151.5
3	400 5	300	101

In the tables shown here the seats are given out in size order of the items in

the table, with provisions about what should happen when there are ties.)

Applying D'Hondt (the European version of Jefferson's method) for a house size of 5, yields 3 seats for A, 2 seats for B, and 0 seats for C. The order of the seat being assigned is shown in the table above with numbers towards the right.

Again, A gets 3 seats, B gets 2 seats, C gets 0.

A, B, and C's exact quotas are: A's exact quota is 2.4969, B's exact quota is 1.8727, and C;s exact quota is .6305. The ideal district 480.6. If we use the modified district size of 400, we get A = 3, B = 2.25, and C = .7575. Rounding down we have A = 3, B = 2 and C = 0 as required above.

Using St. Lägue (Webster) we get:

Divide by:	A	B	C
	1200	900	303
1	1200 1	900 2	303 4
3	400 3	300 5	101
5	240	180	60.6

A gets 2 seats, B 2 seats and C 1 seat.

The total vote here is 2403. A's exact quota is 2.4969, B's exact quota is 1.8727, and C's exact quota is .6305. Rounding fractions up which exceed .5 gives A 2 seats, B 2 seats and C 1 seat, which agrees with the table method approach above.

Consider the apportionment A = 3, B = 2, and C = 0.

Let us use the measure seats per people to measure fairness as it applies to A and C:

$|(3/1200) - (0/303)| = .0025$

Now consider the apportionment A = 2, B = 2, C = 1.

Now compute the seats per people measure as it applies to A and C:

|(2/1200) - (1/303)| = .0016

Hence, we conclude that the apportionment A = 2, B = 2, and C = 1 is more fair than the apportionment A = 3, B = 2, C = 0 for the measure representatives per person. It turns out that if one measures inequity between pairs of states using seats per person as a measure, then one must apportion using St. Lägue/Webster to get the best apportionment.

Here is an example modified in a small way from the 6th edition of Mathematical Excursions, by Tannenbaum and Arnold. We have a parliament with 40 seats, h = 40, and five parties which received 400,000 votes. Thus, the ideal district size is 1 representative for 10,000.

State	A	B	C	D	E	Total
Population	140,200	105,100	65,100	54, 800	34,800	400,000
Exact quota	14.02	10.51	6.51	5.48	3.48	40
Huntington-Hill	14	11	7	6	4	42
Webster	14	11	7	5	3	40
Adusted quota d=10,030	13.978	10.478	6.491	5.464	3.47	
Huntington-Hill	14	10	7	5	4	40

Note that the two apportionment methods agree on all the parties except those for parties B and E. Now, consider the two parties B and E with respect to the measure, seats per people.

When B = 11 and E = 3 we have:

|(11/105,100)-(3/(34,800)| = .00001846

while for B = 10 and E = 4 we have:|(10/105,100) - (4/34,800)| = .00001980

so with this measure the Webster apportionment is fairer than the Huntington-Hill apportionment, since .00001846 is smaller than .00001980.

Now let us compute the minimum of 11/105,100 and 3/34,800. 11/105,100 is .0001047 while 3/34,800 =.0000862 so 3/34,800 is smaller.

So taking .00001846 and dividing it by 3/34,800 we get .2141.

Now let compute the minimum of 10/105,100 and 4/34,800. 10/105,100 = .000095 while 4/34,800 = .000115, so 10/105,100 is smaller.

So taking .00001980 divided by 10/105,100 we get .2081. Now we see that the Huntington-Hill apportionment, measured via relative seats per voter is smaller than that for the Webster apportionment.

To summarize, in this example, Webster is a better apportionment for absolute difference of seats per people while Huntington-Hill is a better for relative difference of seats per voter.

It turns out to be a general theorem that for pairwise equity of parties (states) each of the different methods (Adams, Dean, Huntington-Hill, Webster, Jefferson) is optimal when measured by some absolute measure of equity. However, Huntington-Hill is optimal when measured by all of the relative measures that are optimal in the absolute cases.

Much of the theory here was developed by E.V. Huntington, a mathematician who taught at Harvard University. Huntington had looked into the 32 ways that the inequality pi/pj > ai/aj (where the population of state i, is pi and the number of seats given state i is ai) could be rewritten by "cross multiplication." He worked out the different measures of "inequity" between pairs of states that could be used in this way. He observed that in a comparison between two states who had average district sizes of 100,000 and 50,000 compared with 75,000 and 25,000, the absolute difference is the same. However, he thought that the inequity was "worse" in the second case because 50000/25000 is 2 while 50000/50000 is 1. The relative difference to Huntington seemed a better measure. (Relative difference between x and y being defined as |x -y|/min(x,y).) Of the absolute differences the two most "natural" are |pi/ai - pj/aj| which is optimal when Dean's method is used and |ai/pi -a/pj| which is optimal when Webster's method is used.

For the measure

$$| a_i - a_j (P_i / P_j) |$$

Adams method is optimal.

For the measure

$$|P_j / a_j - P_i / a_i|$$

Dean's method is optimal.

For the measure

$$\left|\frac{a_i / P_i}{a_j / P_j} - 1\right|$$

Huntington-Hill is optimal.

For the measure

$$|a_i / P_i - a_j / P_j|$$

Webster is optimal.

For the measure

$$|a_i(P_j / P_i) - a_j|$$

Jefferson's method is optimal.

To supplement the example worked out above you may want to also look at the analogous work in regard to the example below where the numbers are slightly changed. (These numbers are also due to Tannenbaum and Arnold.) Here, the divisor must be adjusted to complete the Webster apportionment while in the previous version the divisor had to be adjusted to complete the Huntington-Hill apportionment.

State	A	B	C	D	E	Total
Population	140,800	104,800	64,800	54, 800	34,800	400,000
Exact quota	14.08	10.48	6.48	5.48	3.48	40
Huntington-Hill	14	10	6	6	4	40
Webster	14	11	6	5	3	38
Adusted quota d=9965	14.13	10.517	6..503	5.499	3.492	
Huntington-Hill	14	11	7	5	4	40

Above, we looked at fairness in terms of comparisons between two states, and measuring pairwise equity from different points of view (e.g. relative and absolute differences; using different expressions involving claimant populations and claimant seats). However, we could also treat apportionment as a global optimization problem. Compute among all different ways to apportion 40 seats to 5 claimants based on the data above, choose that method that minimizes the sum of the differences in absolute value between the ideal district size (total population divided by total number of seats h) and the size of the district based on the number of seats actually assigned and the population of the claimant state. Of course, there are other measures that could be used for this global optimization. Different measures will typically require that one use different methods.

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