

## Coalition Games and Sharing (Profits) (2020)

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From the time we are little we are encouraged by our parents to share and cooperate. However, anyone who has observed a group of youngsters in a sandbox can see the tension between theory and practice! The same tension exists in many phases of adult society. When businesses cooperate too closely, the public cries "foul," and calls upon the government's anti-trust unit. Yet in international relations, lack of cooperation causes misery for millions and death for others when countries go to war or don't cooperate to control an epidemic.

When we took our first look at game theory we considered matrix games such as the "paradoxical" game below where players who behave rationally by playing a dominating row strategy do worse than if they did something else. (Brief reminder, each player, Row and Column, without consulting each other decide, in Row's case which of two rows to pick to play, and in Column's which of two columns to pick to play; and then get the payoff associated with their actions - If Row plays Row 2 and Column plays Column II, then Row loses 4 and column loses 4, that is, the payoff is (-4, 4) for the pair of players.

	Column I	Column II
Row 1	(8, 8)	(-6, 20)
Row 2	(20, -6)	(-4, -4)

Another way of representing games is to draw a tree which keeps track of the moves that the players make. The 1-valent vertices (leaves) of this tree are the outcomes and have payoffs attached to them. Below, a different way of "representing" certain kinds of games will be looked at which has intriguing applications. This approach is of most interest when there are more than 2 players and involves many ideas from combinatorics and geometry.

How can the benefits of cooperation be attained? By constructing a mathematical model of some of the types of situations where interests of different entities are in conflict, perhaps we can find the route to the benefits of cooperation and anticipate those situations where troubles can arise.

Consider the following plausible but hypothetical situation. Three corporations have been offered space in a new industrial park but have similar businesses. The corporations A, B, and C have prepared (honest?) projections of what profits they can make separately (in millions of dollars) in the new industrial park.

A: 80

B: 100

C: 120

At the meeting of the CEO's of the corporations it is suggested that they investigate the advantages of merging together and of having two or more of the corporations cooperate together. The profits of such joint ventures are determined to be:

A and B: 184

A and C: 248

B and C: 250

A, B, C: 390

Luckily the three corporations have nearly identically sized work forces and approximately equal current economic health, so a possible complication introduced due to such factors can be disregarded. (The difference in profits reflects geographical considerations for the way the corporations operate rather than ones due to work force size.)

How should the three corporations proceed? Since it may not be clear what to do, in time-honored mathematical tradition one might try to consider a simpler case. What would be a sensible way to treat the saving for two corporations rather than three? To get a feel for the problem, consider how the two corporations A and B above might share profits if they worked together. In light of the fact that the corporations have equal workforces it might seem natural merely to split the profits from merging. Adopting this approach would require A to get \$92 and B to get \$92 to share the joint profit

reflecting their joint profit of 184. This is appealing to A but B's profit from acting on its own would be higher than being part of a coalition which collectively gets more than the profit each corporation got separately. How the "spoils" of cooperation get divided matter.

Here are some questions to think about.

1. What is the best way for coalitions to share collective gains?
2. How would you solve the problem of assigning the profits to A and B if that merger took place?
3. How would you take into account the workforce sizes of the two corporations if they were not equal?
4. How would you extend your ideas to the profit-sharing sharing problem for the three corporations?
5. What fairness principles do you think a general profit-sharing scheme should obey?

Among the intriguing features of this problem for teachers are the ease of getting started, being able to draw on students' experiences, the variety of approaches possible, and related problems which offer subtly different challenges.

Abstracting from the kind of situation we saw above it is traditional to list the "value" that each coalition of players in a game can attain for that coalition, and use set notation to represent the coalitions and  $v(S)$  where  $S$  is a set to give the value of the coalition. (Note that usually one writes  $V(S)$  rather than  $v(\{S\})$  though sometimes the set bracket is used for emphasis that one is talking about a set.) Intuitively, the idea is the players of  $S$  can guarantee themselves  $v(S)$  units if they work together. It is important to remember that this does not discuss how the players might "divide the spoils" of working together that they get, that is, the  $v(S)$  units that the coalition can earn. There are many ways the spoils (profits, costs) might be shared, which include equal division but this is not required. Games of this kind belong to what are called TU games, TU for transferable utility. Thus, the payoff in a TU game is one that the players can transfer among the themselves because they value the payoff on the same scale. Even though money can have different "utility" to different players it is the most common "prize" in TU games. TU games are typically considered part of cooperative game theory, as compared with non-cooperative games, where the players

can't work together even though this might benefit them.

Sometimes the players of such a game are given numbers as names, thus, players 1, 2, 3 and 4 and sometimes letters, players A, B, C, and D, for the 4 players of a game. Often the set of players is denoted by N. Since each player can be in a coalition or not, if there are n players we need to record values for  $2^n$  sets or coalitions. (In set theory, what we are talking about is the power set of the set of players that must be assigned "values.") In this framework the coalition game above can be written:

$v(\{\}) = 0$  (the set of no players - the empty set - can't get anything of value!)

$v(\{A\}) = 80$ ;  $v(\{B\}) = 100$ ;  $v(\{C\}) = 120$

$v(\{A,B\}) = 184$ ;  $v(\{A,C\}) = 248$ ;  $v(\{B,C\}) = 250$

$v(\{A,B,C\}) = 390$

The last entry above is the payoff for what is called the grand coalition. In cooperation situations it is often of interest to make sure the grand coalition forms but the issue of how this coalition divides its "value" has not been discussed yet.

So why would players A and B work together in some situation? Presumably, it would be if in situations involving costs, say, they would have lower costs by working together, or in cases of profits, they would have higher profits by working together. Though above we motivated our discussion in terms of profits, in many cases in everyday life it is cost sharing problems that are more natural. In a profit-sharing situation where  $v(S)$  represents profit to the coalition S, it seems reasonable to require that  $v(S \cup T)$  (I will use "capital U" for the union symbol below), exceeds or equals the sum of  $v(S)$  added to  $v(T)$ . In symbols  $v(S \cup T) \geq v(S) + v(T)$  (where S and T have no elements in common). When coalitions of the power set (all subsets) are given values subject to this condition, one refers to the characteristic function representation of the game. A game in characteristic function form is said to be constant sum, if for every subset of players (denoted by N):

$$v(S) + v(N - S) = v(N)$$

Here  $N - S$  denotes the set of players not in S, and  $v(N)$  is another way of referring to the payoff to the grand coalition, N.

For games with many players the intersection of two coalitions may not be the empty set, the set with no players. One natural assumption one might want to have hold for games where one hopes to improve one's gains is:

$$v(S \cup T) \geq v(S) + v(T) - v(S \cap T) (*)$$

The reason for this is that if there is an element in  $S \cap T$  (this is the intersection of the two sets  $S$  and  $T$ , that is, the players who are in both the set  $S$  and the set  $T$ ), then their "value" will get counted in both  $v(S)$  and  $v(T)$ ; it seems to make sense to subtract this double counting off in understanding if there is incentive for coalitions corresponding to sets with overlapping members to form.

A coalition game is called CONVEX if (\*) holds for all coalitions  $S$  and  $T$ .

If only

$$v(S \cup T) \geq v(S) + v(T)$$

holds, the game is said to obey the "superadditivity" condition. From a theoretical point of view superadditivity may or may not hold, and a game may be convex or not. In some "real world" situations it seems natural that these conditions might or might not hold, but in some contexts it seems natural to expect that these conditions would hold. For profit sharing, unless by joining forces two coalitions can raise their "joint" profits, there seems to be no incentive for them to cooperate. Usually, there must be incentives for cooperation to occur.

If the grand coalition forms how should it "divide" the spoils? Presumably we are interested in the results satisfying two things:

- a. The splitting of the pot between the members of the grand coalition be fair, that is, reflect the structure of what profits the players would have if they did not form the grand coalition or something happened other than the grand coalition.
- b. The coalition be "stable," that is, no member of the grand coalition be tempted to leave the coalition because the share of the earnings of the grand coalition that it gets are not pleasing, and that some smaller coalition to work with gives them greater benefit.

In order to get further it is convenient to think about the payoff that each player  $i$  gets in a coalition game. (Here it helps to think of the players as being

named with numbers instead of letters. For the example above we could take A to be 1, B to be 2 and C to be 3).

Let  $x(i)$  denote the payoff given to player  $i$  in the coalition game being studied. There are some obvious sensible assumptions on the values that would make  $x(i)$  values acceptable.

a. For each player  $i$ ,  $x(i)$  is at least as large as  $v(\{i\})$ .

No player will join a coalition without getting at least as much return as he/she would have playing alone. This condition is sometimes called individual rationality.

b. The sum of the values of the  $x(i)$  over all players  $i$  is the value of the grand coalition.

This condition resembles "Pareto optimality" in that it makes sure that all of the utility comes to the group collectively.

c. One can also ask that for every coalition the payoff to the members of this coalition be at least as much as they can get by participating in that coalition.

If full cooperation is to be achieved, what the players make individually by being in the grand coalition should be the value that the grand coalition can obtain. Otherwise, some of the "total" utility to the group is not being "returned" to the players in the full group of players.

For games of this kind, the term imputation is the concept of what might be "reasonable" way to divide the winnings from coalitions that arise, though some scholarly sources define this term in different ways.

Another important concept here is that of the core of a coalition game. The core of a game is that collection of ways of distributing the utility of the grand coalition that are feasible (they can be "achieved") and that cannot be improved upon by some smaller coalition. Thus, there is no incentive to try some other "division" of the spoils. The core may have many points, a single point, or be empty. When the core is empty it is usually very difficult for the players to cooperate. When the core is non-empty but not unique there are many ways to split the pot but at least there is some beneficial way to split it from each individual player's point of view, and those of the different possible coalitions. The core was apparently first thought of by D. B. Gillies (1953) but studied in more detail by H. Scarf and Lloyd Shapley.

For convex games the Shapley value must be in the core but for more general coalition games it may not be in the core. Games which have constant sum have an empty core.

Example: (Warning: This example does not obey all of the conditions above.)  
What might you expect to happen?

$$v(\{\}) = 0$$

$$v(\{A\}) = 80; v(\{B\}) = 100; v(\{C\}) = 120$$

$$v(\{A,B\}) = 190; v(\{A,C\}) = 190; v(\{B,C\}) = 230$$

$$v(\{A,B,C\}) = 270$$

Reference:

H. Peyton Young, *Equity in Theory and Practice*, Princeton U. Press, Princeton, 1994.