Weighted Voting Examples (2020)

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Weighted voting is useful for conducting votes in legislative bodies where the players deserve unequal influence. Examples include the US Electoral College, the European Union and county governments in New York State. In the Electoral College the number of votes a state casts is the number of its members in the House of Representative plus 2, the number of senators from the state. Though not widely known, although the District of Columbia is not a state and has no representatives in Congress, it does cast 3 votes in the Electoral College, so the total number of votes cast is 435+100+3=530. In order to be elected President, a candidate must get the vote of 270 electors.

Suppose we have a weighted voting game where a "bill" can pass with a simple majority. In Example 1, there are 4 players named 1, 2, 3, 4 who cast weights of 5, 4, 2, and 1, respectively. Half the sum of the weights rounded up to the next integer suggests a quota for action by a coalition be set at 7. The way this situation is usually notated is shown in Example 1. If weights were chosen proportional to population, it is easy to see examples where POWER or INFLUENCE is not proportional to WEIGHT. Thus, a player can have positive weight but be a dummy, that is, not be a member of any minimal winning coalitions. A coalition is minimal winning if its members cast weights which sum to more than or equal to the quota, but any subset of these players does not have this property. So what one wants to be able to do is to set weights so that POWER/INFLUENCE in the legislature is proportional to the population. The only issue is that for many weighted voting games there are DIFFERENT ways to measure the power for the same game. How can one tell which is a better way of measuring power for the same game? This is analogous to what we saw happen in elections. In deciding an election based on a particular set of ballots, different election methods yield different

winners. Often, for the same weighted voting game, the Coleman, Banzhaf and Shapley/Shubik power indices give different values for the power of the players, and it is not clear which power index gives the players a better view of the influence they truly have in a given voting situation. The series of examples below shows the "pattern" of the way that the collection of minimal winning coalitions varies as the quota changes but the weights are kept fixed.

Example 1:

Weighted voting game: four players 1, 2, 3, 4

[7; 5, 4, 2, 1]

The minimum coalitions are:

{1, 2}, {1, 3}, {2,3,4}

Clearly there is no player who always gets his/her way (dictator), and furthermore there is no *veto* player because no player is a member of every minimal winning coalition.

In some voting situations different kinds of bills require more support to pass. For example in the US Congress, a bill to amend the Constitution requires more than a simple majority. What happens in a weighted voting game such as that of Example 1, when the quota goes up? Not surprisingly, there may be fewer or altered composition for minimal winning coalitions.

Example 2:

Weighted voting game: four players 1, 2, 3, 4

[8; 5, 4, 2, 1]

Note: weights are the same but the quota has increased by 1.

There are now only two minimal winning coalitions.

{1, 2}, {1, 3, 4}

Perhaps not surprisingly the structure of the minimal winning coalitions has

gone down and there are now only 2 minimal winning coalitions as compared with three when the quota was 7. Also, now Player 1 is a veto player; without player 1's support no action can be taken.

Example 3:

Weighted voting game: four players 1, 2, 3, 4

[9; 5, 4, 2, 1]

There is only one minimal winning coalition:

{1, 2}

So in this situation Players 3 and 4 are "dummies" because they have no "power." Both Players 1 and 2 are now veto players, but there is no one player who can take action on his/her own. Any reasonable power index should in this case assign Players 1 and 2 equal power despite not having equal weight and should assign Players 3 and 4 zero power.

Example 4:

Weighted voting game: four players 1, 2, 3, 4

[10; 5, 4, 2, 1]

The minimum coalitions are:

{1, 2, 3}, {1, 2, 4}

Note that Players 3 and 4 are no longer dummies and the number of minimal winning coalitions has increased! The number of minimal winning coalitions is not monotonically decreasing as the quota goes up.

Activity 1:

1. Compute the value of the Coleman, Banzhaf, and Shapley power of the 4 different games above. Which of the three power indices do you feel reflects YOUR intuition about which power index is better?

Comment: Note that these three power indices do not "directly" give people who are in SMALLER minimal coalitions more power. Other power indices do

give more power to players in small winning coalitions since it seems as if it might be easier to form a coalition with 2 players than one with 5 players. The Packel-Deegan power index takes coalition size into account in assigning power to the players but it is not clear if this more accurately describes the "real world" or not.

It may seem strange to look at games where the quota is not more than half the sum of the weights but there are some voting situations where this happens. For example, for some cases to be heard by the US Supreme Court (which at full strength has 9 members), it is not necessary that at least 5 justices agree to hear the case; in some situations if some 4 justices want the case heard, then the whole Court will get to hear the case.

Activity 2

For the game:

[Q; 5,4,2,1] as Q varies from 1 to 12 (which gives rise to 12 games)

a. List the minimal winning coalitions.

b. List the Coleman, Banzhaf, and Shapley/Shubik powers for these games.

c. Look for patterns in the results you see.

Repeat this for other games with different weights and 4 players and with other games for 5 players.

When we have a fixed number of players with the same or differing weights, sometimes the resulting minimal coalitions as we change the quota stays the same and sometimes the minimal wining coalitions change. So from the point of view of "theoretical mathematics," there are interesting questions to ask.

Research Question:

1. For the situation with a fixed number n of players and the weights for players 1 to n don't change, how many inequivalent (non-isomorphic) games can there be as n varies?

2. Some voting games can be described just by specifying the minimal winning coalitions with the condition that supersets of a winning coalition also be winning. When can such games be represented as weighted voting games?

Thus, the UN Security Council is specified in terms of winning coalitions but it turns out one can assign weights to the players so that one can think of it as a weighted voting game.

3. Which weighted voting games are such that the power of the players if one measures voting power by:

a. Coleman Power Index

b. Banzhaf Power Index

c. Shapley Power Index

have weights which are proportional to the power index values?

Example:

[2; 1, 1, 1] is an example where the power of the players is proportional to the weights of the players, but this example is rooted in the high symmetry of the minimal winning coalitions. Are there other examples?

For example one might start with a voting game, compute the Banzhaf Powers where all of the powers were written using a common denominator. Now, can one use the numerators as weights, and for that game recover the same power values that one started with?

4. Are there games whose properties can be described easily where the different power indices result in the same values?

Comments:

1. While conceptually Banzhaf and Shapley/Shubik Power are not that hard to grasp, computationally they require more effort than is reasonable to do by hand (my favorite choice for insight). For 4 players to do a Banzhaf calculation by hand requires a table of 16 lines (2^4) and a Shapley calculation by hand requires a table of 24 lines ($4!=4(3)(2)(1)$). For 5 players the equivalent numbers are 32 lines and 120 lines. There is software to do the calculations and there are tables where one can find counts of the number of inequivalent voting and weighted voting games with relatively small numbers of players. Some games which can be specified by lists of minimal voting coalitions can not be represented with weights, so the counts for voting games and weighted voting games are not the same.

2. There are mathematical, political science and behavioral issues all at play here. In particular, in actual weighted voting games there is a big question about the empirical nature of the coalitions that actually form, and perception of the people who play these games as to the power/influence they exert.

References:

Brandt, F. and V. Conitzer, U. Endriss, J. Lang, A. Procaccia, Handbook of Computational Social Choice, Cambridge U. Press, Cambridge, 2016.

Taylor, A. and W. Zwicker, Simple Games, Princeton U. Press, Princeton, 1999.