Nash Equilibria for 2x2 Non-Zero Sum Games (2021)

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The two most famous "paradoxical" 2x2 (two players, two actions for each player) games are Prisoner's Dilemma and Chicken. In Prisoner's Dilemma dominating strategy analysis can lead both players to negative payoffs even though there are actions which lead both to positive payoffs. Prisoner's Dilemma is actually a family of games where the numerical values of the payoff vary and the different versions have different "feels" to them. So when all payoffs in a Prisoner's Dilemma problem are positive such a version of the game has a very different flavor from when some of the payoffs are negative.

Consider this version of Prisoner's Dilemma

Prisoner's Dilemma

	Column I	Column II
Row 1	(6, 6)	(-10, 30)
Row 2	(30, -10)	(-3, -3)

Row is better off playing Row 2 whichever column the Column player plays. Thus, 30 is better than 6 and -3 is better than -10. The symmetry of the game means that Column reasons the same way, so one can reason Row should play Row 2 and Column should play Column II and both players have outcomes of -3, far worse than the "irrational" outcome of 6 units for each player. Using a motion diagram analysis one sees that the only equilibrium (Nash equilibrium) in this game is the (-3,-3) outcome. Playing Row 1 by Row, and Column I by Column is a Pareto Improvement over the Nash equilibrium. Thus, the theory of the Nash equilibrium is appealing but in practice playing games to get to a Nash equilibrium may not seem sensible.

A Nash equilibrium represents actions the players can take so that if some player "deviates" from the equilibrium action that player gets a worse (or no better) payoff.

Nash's Theorem says that for very general classes of games there will always be:

a. An equilibrium in pure strategies

or

b. An equilibrium in mixed strategies

or

c. Both of the above

One can think of having an equilibrium in mixed strategies as "including" the case of pure strategies. One takes certain actions with probability 1 and others with probability 0.

Nash's proof of this result is as an existence theorem and uses the Kakutani fixed point theorem. To give you the flavor of a fixed point theorem, a simplified version of the Brouwer Fixed Point Theorem says that for n-dimensional Euclidean space, a continuous function of a bounded and closed (thus, compact) and convex set onto itself has a fixed point (for some z, $f(z) = z$).

Let us see if there is a mixed strategy equilibrium for the version of Prisoner's Dilemma above.

If Row plays Row 1 with probability p and Row 2 with probability (1-p) for what choice of p will this equalize Column's earnings when Column always plays Column I versus when Column always plays Column II?

This would mean that

$$6p + (-10)(1-p) = 30p + (-3)(1-p)$$

Solving this linear equation we get:

$6p + 10p - 10 = 30p - 3p - 3$

$11p = -7$

$p = -7/11$

But p is a probability and can't be negative, so this means there is no mixed strategy equilibrium for this game.

You can try to see if there might be different values for the numbers in the matrix above which would make it possible for there to be mixed strategy equilibria.

Chicken

	Column I	Column II
Row 1	(3, 3)	(-6, 4)
Row 2	(4, -6)	(-7, -7)

Chicken is often used as a model for political games such as the Cuban Missile Crisis. Note in this game there are NO dominating rows or dominating columns. However, using motion diagram analysis we find two Nash equilibria. at (-6,4) and (4,-6). However, the players might be able to find their way to playing Row 1 and Column I in many repeats of this game, even though this outcome is not stable, not an equilibrium.

Does this version of Chicken have a mixed strategy equilibrium?

From Column's point of view he/she would like to equalize the results of what happens whichever Row plays Row 1 all of the time or Row 2 all of the time. So with Column playing Column I q percent of the time and Column 2 (1-q) percent of the time, we have

$3q + (-6)(1-q) = 4q + (-7)(1-q)$

$3q + 6q - 6 = 4q + 7q - 7$

$2q = 1$

$q = 1/2$

With q equal to 1/2 the payoff to Row would be on average -3/2. This can be found by, say, substituting the value of q in the first equation above: 3(1/2) - 6(1-1/2) = 3/2-3 = -3/2.

By symmetry, Row can equalize the results of Column playing Column I or Column II. So there is another Nash equilibrium for Row playing each of his/her row actions 1/2 the time and Column playing each of his/her actions 1/2 of the time. The resulting payoff is -3/2 to each of the players. The "unstable" outcome of (3, 3) from Row playing Row 1 all of the time and Column 1 playing Column I all of the time is still more attractive. However, even if there is a treaty agreeing to play in this way, each player might be tempted to break the treaty on the next play of the game because that way the treaty breaker would get a "bonus" while "hurting" his/her opponent.

Also note there is another possibility when the size of the entries in a Prisoner's Dilemma or Chicken matrix are "just right."

Suppose in the Prisoner's Dilemma game above Row agrees to play Row 1 on odd numbered plays of the game (e.g. 1st, 3rd, etc.) and Row 2 on even number plays of the game (e.g. 2nd, 4th, etc.) and Column agrees to play Column II on odd numbered plays of the game and Column I on even numbered plays of the game.

Here are the cumulative earnings:

First play: Row -10, Column 30
Second play: Row 20, Column 20
Third play: Row 10, Column 50
Fourth play: Row 40, Column 40
etc.

Thus, if Row and Column can continue to work together they can do a lot better than each of them playing Row 1 and Column I all of the time.

Unlike the situation for zero-sum games, non-zero-sum games are much more complex.

Another issue for games in general that is important is to what extent it is possible to realistically get numbers to put in the payoff matrices. Clearly, the players in political games will rarely know what their payoffs and the payoffs of their opponents might be. One might be able to rank the outcomes with ordinal values rather than cardinal values, but this makes it less "precise" to meaningfully analyze the game. Similar issues are involved with decision making, games against nature. Mathematics can suggest tradeoffs between different strategies but rarely can say what to do.