# Notes for Remote Presentation 1:

# Game Theory/Fairness Modeling

January 11, 2021

One can't teach (convey) to others what one does not know oneself.

Hence, I adopt a breadth over depth approach to curriculum!

Many people find it easier to learn mathematical ideas when "told" about them rather than reading about them in a book. Learning new mathematics is HARD.

Much of what you learn this semester can be used as contexts and modeling examples for topics in traditional K-12 curriculum public school curriculum.

For example, I will show you some interesting applications of working with fractions and solving first degree (linear) equations or two equations in two "unknowns" - standard algebra topics

# What is "mathematical game theory?"

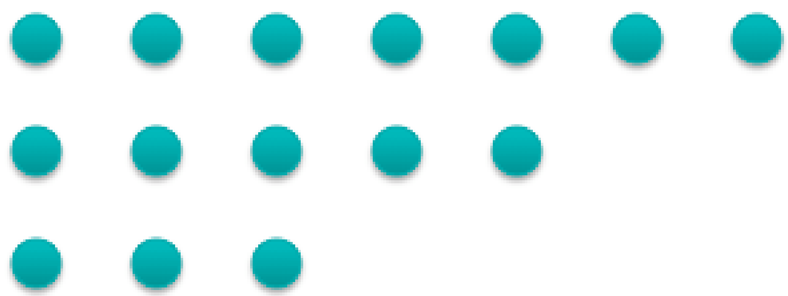
Mathematics has two major views about approaching "games":

First point of view:

* combinatorial games

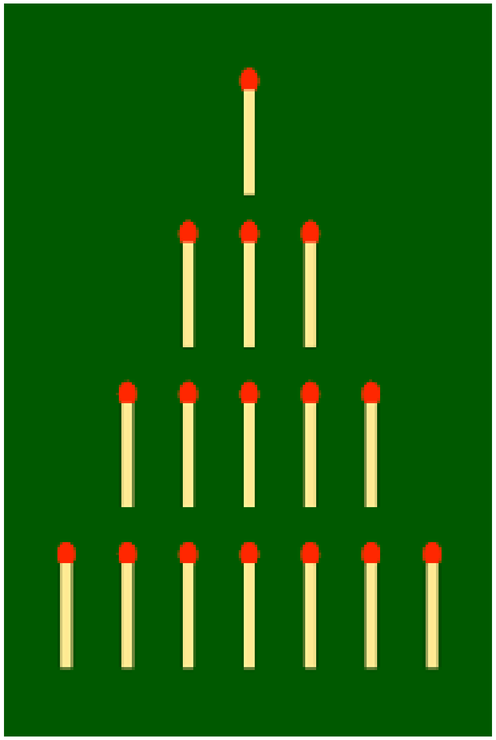
Some examples:

* Nim (select sticks from piles (heaps of sticks or stones) (pieces are the same for both players - impartial games)



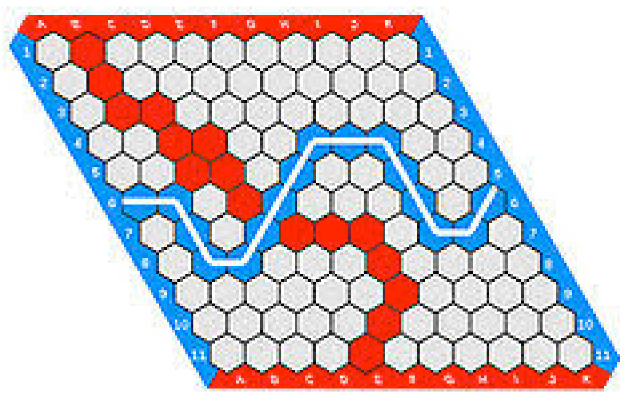
(You can think of this as two different Nim positions. One with 3 "heaps," the rows, the other with 7 "heaps." )

Another Nim position (there are 4 rows (heaps) with 1, 3, 5, and 7 items in these rows).



(From Wikipedia)

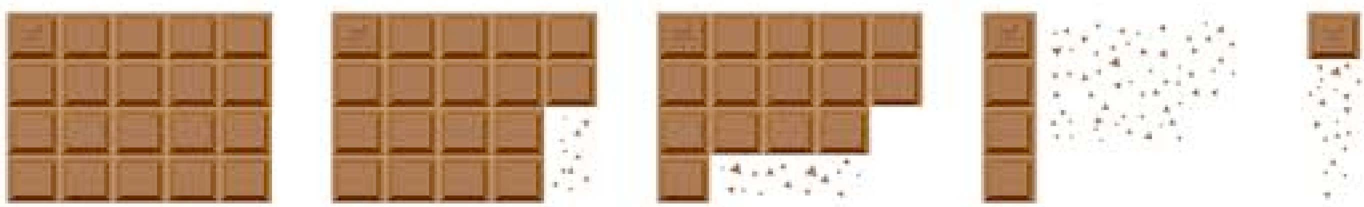
* Hex (partisan game - each player has their own pieces: one player is red, the other blue) Goal: make a red or blue chain between your two sides of the board.



(From Wikipedia.)

# *Chomp

Move: pick a square of chocolate and bite off what is down and to the right.



(from Wikipedia)

Goal: If possible by looking at the initial board and the rules, who will "win" the game and how should a person play to achieve that win? If the position is losing how can one "drag out" one's loss? One can hope one might win because one's opponent makes a mistake! *Solving the game* means finding optimal play for each player.

John Horton Conway (1937-2020) (One of the greatest mathematicians of recent times)

Termination rule - so called "normal" play:

*If you can't move you lose!!*

If you can't move you win is called the misère version of the game. For reasons not well understood misère games are much harder to solve than "normal play" games.

The "classic book" on combinatorial games is:

**Winning Ways for Your Mathematical Plays** by Elwyn Berlekamp, John Conway and Richard Guy (now all deceased)

# Other viewpoint on games:

* conflict issues drawn from economics (business), political science, philosophy and psychology

*Prisoner's Dilemma

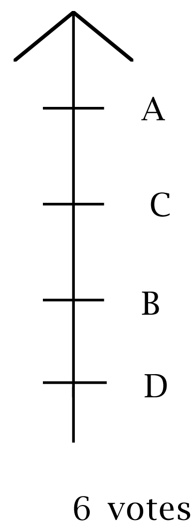
*Chicken

(models of confrontation situations - labor vs. management)

Individuals who call themselves game theorists, whether they are mathematicians, economists, political scientists, philosophers or psychologists often study:

* elections and voting

preference or ranked ballet:



Counting such ballots takes as inputs voter preference ballots and outputs a "winner" or collection (set) of winners. Hence, it is a non-standard example of a function!

* apportionment

After the 2020 Census is completed how many seats of the 435 in the US House of Representatives will each state get?

# * cost sharing

Two towns have stand-alone costs to build newly required sewage treatment plants. Building a plant together saves money. How might they share what they save by working together?

* allocation of scarce resources

Recent examples:

Covid-19 inspired issues:

Ventilator allocation to hospitals:

State X has been given 2000 new ventilators. How many should be assigned to the different hospitals (NYS has 215) in the state?

Distribution systems for COVID vaccines.

What is a fair way to distribute a highly desired but scarce resource? a new vaccine; water from the Colorado river?

* settling estate claims

* bankruptcy settlements

*school choice

* auctions

* matching markets

* fair division

Example: How to divide property after a divorce.

* weighted voting

When Britain left the European Union (EU) ("brexit") what changes in "voting" might make sense?

Remember that Luxembourg and Germany are both members of the European Union - should they have the same influence in decisions made by the EU?

One way to deal with such situations uses *weighted voting*. More important, more populous, more economically powerful countries would cast MORE votes. This contrasts with the idea of "one country, one vote." Countries can cast different numbers (blocks) of  vote.

In the United States:

Senate: Each state gets two senators.

House of Representatives: Number of representatives is proportional to population.

This represents two different views of equity or fairness!

* entity equity

* proportional equity

We will learn about many different approaches to being "fair."

# Matrix game:

# Matrix is a fancy word for a table:

| | | | | |
|---|---|---|---|---|
| 2 | -4 | 5 | 0 | -2 |
| -4 | 6 | x | 7 | -3 |
| 7 | -1 | -4 | y | -5 |

The matrix above is 3x5; three rows and 5 columns.

The first number is always the number of rows.

**Matrix games:** (This game is 2x2)

Moves for player named Column

| Moves for player named Row. | | Column I | Column II |
|---|---|---|---|
| | Row 1 | (3, -3) | (-2, 2) |
| | Row 2 | (6, -6) | (2, -2) |

Row can make two moves (actions) as can Column. One play of the game can lead to one of four outcomes.

If you had play this game once how would you play?

If you had to play this game many times how would you play?

Is this game fair?

# How would you play?

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (-1, 1) | (7, -7) |

The first number in the pair is the payoff to Row and the second number is Column's payoff.

Note that the sum of the payoffs is zero for each choice of actions of the players.

This is a zero-sum game!!

It is traditional for zero-sum matrix games to put only one number in each cell of the matrix, and that number is the *payoff from Row's point of view!*

| | Column I | Column II |
|---|---|---|
| Row 1 | 3 | -2 |
| Row 2 | -1 | 7 |

# Is this game fair?

# How would you play?

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (0, 0) | (1, -1) |

Is this game fair?

How would you play?

Note: 0 can be the payoff for both players.

| | Column I | Column II |
|---|---|---|
| Row 1 | (1, -1) | (-1, 1) |
| Row 2 | (-1, 1) | (1, -1) |

By symmetry it might seem this is a fair game! Payoffs are symmetric for the players. (Properly played this is a fair game.)

In the next game note that there is one payoff which is very large (large gain or loss) compared with the other outcomes.

| | Column I | Column II |
|---|---|---|
| Row 1 | (100, -100) | (-10, 10) |
| Row 2 | (-10, +10) | (1, -1) |

Is this game fair? It will turn out the answer is "yes!"

# How would you play?

What would it mean for games like the ones we just looked at to be fair?

First, let us look at some ideas about how to play zero-sum matrix games.

Moves for player named Column

Moves for player named Row.

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (6, -6) | (2, -2) |

Sometimes no matter what your opponent does, some action (move) is better than any other choice.

Moves for player named Column

Moves for player named Row.

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (6, -6) | (2, -2) |

# For Row: whatever Column does, row 2 is better than row 1! (6>3; 2>-2)

Moves for player named Column

Moves for player named Row.

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (6, -6) | (2, -2) |

# For Column: whatever Row does, column II is better than column 1! (2>-3; -2>-6)

So it makes sense, for no matter how many independent plays are made of this game, for Row to always play row 2 and Column to always play column II. Payoff every time is: Row wins 2, Column loses 2. The game is UNFAIR: Row always wins, Column always loses when both play OPTIMALLY!

If the players of the original game are playing "rationally," it is as if they were playing the following 1x1 matrix game:

| | Column II |
|---|---|
| Row 2 | (2, -2) |

A dull game to play especially for Column, who always loses.

Note that Row moves by picking a row to "play." Column moves by picking a column to play.

When one finds a row that dominates another row one can get a SMALLER game matrix by CROSSING out the row which is DOMINATED, leaving the dominating row intact.

So a first step in analyzing how to play a zero-sum matrix game is by looking for rows or columns that might dominant other rows or columns.

Note: Initially there may be NO dominating row but after eliminating a dominated column, there may be a dominating row.

Initially there may be NO dominating column but after eliminating a dominated row, there may be a dominating column.

Thus, Row (the row player) looks for dominating rows.

Thus, Column (the column player) looks for dominating columns.

Simplify this **zero-sum** game matrix as much as possible. Payoffs are from Row's point of view. A payoff of -4 is a GAIN for Column and a loss for Row.

| | I | II | III |
|---|---|---|---|
| 1 | 2 | 0 | -9 |
| 2 | 3 | 1 | -7 |
| 3 | 1 | -4 | 2 |

What does one do if dominating strategy analysis does not simplify the game matrix of a zero-sum game?

Let us take a break to see how a geometric tool can help with insight into game theory. It will be useful to you in many other contexts.

Dots and lines diagrams known as:

*graphs

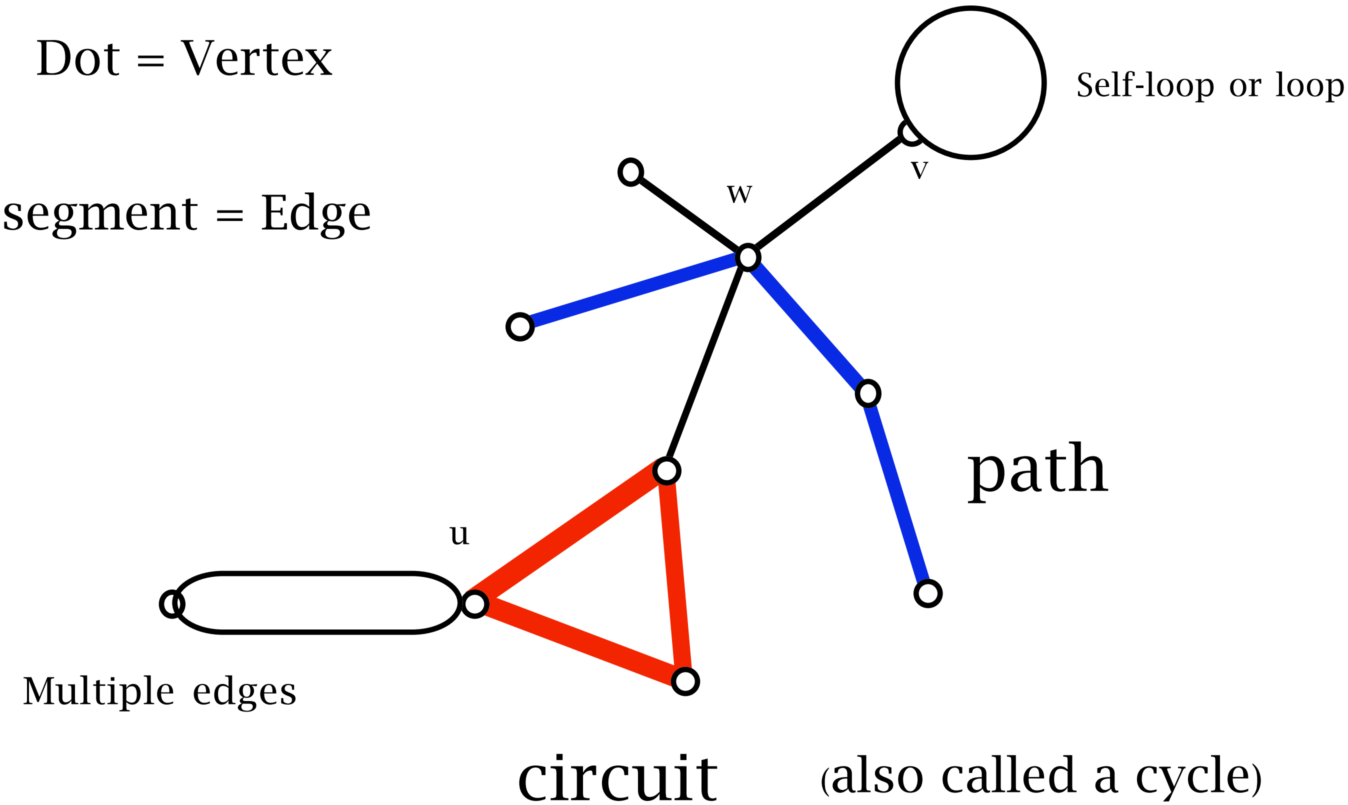
* digraphs (directed graphs) (arrows on the lines)

This topic belongs to the areas of mathematics known as discrete mathematics, combinatorics, discrete geometry and graph theory.

# Introduction (primer) of graph theory:

Dot = Vertex

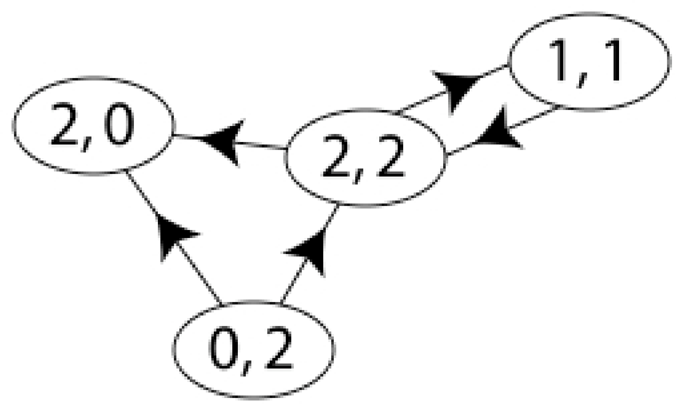
Line segment = Edge




This graph has 10 vertices and 12 edges.

The valence or degree of a vertex in a graph is the number of (local) line segments which meet at the vertex. The valence of v is 3, of w is 5, and of u is 4.

Digraph: 4 vertices; 5 directed edges.

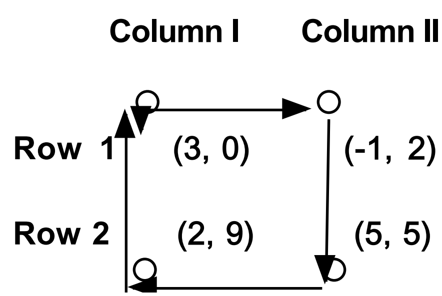



Numbers show indegree and outdegree of vertices.

Indegree: number of directed edges coming into a vertex

Outdegree: number of directed edges leaving a vertex

Example: Motion diagram of a 2x2 matrix game - in this case the payoffs are not zero-sum:




One dot for each payoff.

This diagram shows that there is no outcome that is STABLE because one of the players has an incentive to change his/her actions. A "stable" outcome would be one with OUTDEGREE zero.

# This game has no dominating rows or columns: How would you play this game?

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (-1, 1) | (7, -7) |

Suppose you were required to play this game 100 times - a new round after each prior round is completed.

How would you decide what sequence of moves to make?

Suppose you are Column and you notice that Row always plays this pattern of rows:

1, 1, 2, 1, 1, 2, 1, 1, 2, ....

# What would you do?

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (-1, 1) | (7, -7) |

Row's pattern:

Rows: 1, 1, 2, 1, 1, 2, ...

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (-1, 1) | (7, -7) |

If Row said I plan to play Row one for sure, what would Column do?

Column can be sure of getting a positive payoff by playing column II!

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (-1, 1) | (7, -7) |

If Row says I plan to play Row 2 for sure, then Column can be sure to win by playing column I.

Thus, if either player detects a pattern in the play of their opponent, they can exploit that information to get BETTER outcomes!

# Thus, for this game:

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, -3) | (-2, 2) |
| Row 2 | (-1, 1) | (7, -7) |

optimal play for both players requires the use of a randomization device!!!

Fact:

For many zero-sum games that are played repeatedly, the best way to play is NOT deterministic but using some way to make one's choices randomly!

But there are infinitely many ways of playing randomly. Is there one which is best?

Answer: Yes! We will see how to find that BEST way of playing randomly.

# Review of randomization and probability theory!

When one does an "experiment" there is a collection of possible outcomes, assumed to be finite for our purposes.

Example: Toss a coin:

Outcomes: head or tail

Example: Roll a standard die (plural is dice)

Outcome: 1 to 6 "spots"

Some coins are "fair" and some are "biased."

Some dice are "fair" and some are biased.

A fair coin will show heads and tails in approximately equal frequencies. Thus if one tosses a FAIR coin 20 times, one will get approximately 10 heads and approximately 10 tails, the sum being 20.

However, and this is a sticking point for most beginners, when one tosses a fair coin 20 times the chance of getting exactly 10 heads and 10 tails is very low!!

In fact the chance of this happening is:

.0000009765625

About one in ten thousand experiments!

We need to understand the difference between the probability of something happening, and the EXPECTED value of an event.

Given the set $S$ of n outcomes of an experiment, called the sample space of the experiment:

$$S = \{ o_1, o_2, o_3, \ldots, o_n \}$$

# Probabilities of events obey:

1. $p(o_i) \geq 0$ (probability of each event is positive or 0.)

2. $p(o_i) \leq 1$ (probability of each event is at most 1.)

3. The sum of the probabilities of all the sample space events adds to 1.

Note: The standard symbol for summation in mathematics is the Greek letter "capital" sigma:

$$\Sigma$$

Thus:

$$\sum p(o_i) = 1 \text{ (summed from } i = 1 \text{ to } n\text{)}$$

# Have an enjoyable week!

If you have questions email me:

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class web page:

https://york.cuny.edu/~malk/gametheory/index.htmlr