Notes for Remote Presentation 11:

Game Theory/Fairness Modeling

April 5, 2021

Bankruptcy problems

Bankruptcy model:

The origins of the bankruptcy model are many but include both recent and ancient ancestors:

a. A parent has willed various amounts in his/her estate but the size of the estate E is not large enough to cover all the amounts!

b. A company has gone bankrupt but there are not enough remaining assets to pay all of the persons/companies owed money.

c. Water sharing arrangements among countries (or states) can't be met because there is not enough water in the river.

d. Disaster relief funds

After floods, tornados or a hurricane the amount a government allocates to help "survivors" will not cover all their claims.

Historical note: (Problem that attracted biblical scholars - examples from the Talmud):

	Claim	100	200	300
Estate	*100*	33 1/3	33 1/3	33 1/3
	200	50	75	75
	300	50	100	150

Row 1: Maimonides gain
Row 3: Proportionality
Row 2: ???? (Perhaps a copying error?)

Important recent contributors to this work:

a. Barry O'Neill (political scientist)

b. Michael Maschler (deceased:1927-2008)

c. Robert Aumann (won Nobel memorial prize in economics - 2005 - shared the prize with the game theorist Thomas Schelling) Majored in Math at City College; MIT Ph.D.

d. William Thomson (economist)

Abstract version:

We have claimants (players) 1, 2,...,k whose claims $c_1,...,c_k$ sum to more than a positive amount E. How much should be given to each claimant?

The assumption is made that the claims are valid (honest).

I usually label the players (letter names) so that claims are increasing in size, but not everyone does this.

Comment: Some claimants may be "richer" than others and if they get a small part of their claims they will not be "hurt" badly. Some claimants may go "bankrupt" themselves if they don't get back what they are owed, even if this amount is small!

This model ONLY looks at the size of the claims and not the "affluence" of the claimants.

The estate E is usually **money** but it could be:

a. water (from a river; reservoir)

b. an amount of a limited medical supply (not enough antibiotics or vaccine doses for all those who might benefit from it)

c. time (access to a time share)

d. raising taxes from different income classes (many poor; few rich- but in America wealth inequality is growing)

The idea here is how to distribute the burden of running a government over the different income classes in the society "fairly."

We saw many ways to decide an election in an appealing manner; many ways to solve a bankruptcy problem appealingly; many ways to solve an apportionment problem in an appealing manner. However, picking which appealing approach is not so clear, and in may cases there are short lists of fairness properties so that none of these appealing methods obey ALL of the items on a short list of fairness properties! (Arrow's Theorem; Satterthwaite-Gibbard; Balinski-Young; impossibility results due to William Thomson for bankruptcy problems.)

Bankruptcy Model:

Fairness axioms for solution methods.

Typical example:

Claims: E is available to distribute

A = 30 B = 90 C = 120 ; E = 150

Fairness axioms for a method:

1. If two claimants have equal claims they should get the same amount.

2. If more money is found to pay off the estate (E grows to a bigger E*) then what a claimant is given should not go down. (Monotonicity in estate size.)

3. Two problems differ in that one claimant Z is asking for more, the other claimants and E (estate size) are unchanged.

Z's share where her claim went up should not now be SMALLER for a fair method.

(Monotonicity in claim sizes.)

Another interesting idea:

Suppose there is a bankruptcy problem solved by method M, and M gives a subset X of claimants D dollars. Now look at the new problem where one uses the amount D to settle the original claims using D instead of E.

Method M is called *consistent* if the members of a subset of claimants X get the same amount when M is applied to X using D as the "new" estate size as when it used E as the estate size.

The Talmudic method is *consistent*. When applied to 2 claimants it gives the same result as concede and divide.

Other methods are consistent as well.

There are many papers concerned with exactly what axioms a method to solve a bankruptcy problem must hold in order for one to be *forced* to use a particular method.

Axiomatics is important in social choice as well as geometry and algebra.

What appealing methods should one use to settle claims "fairly?"

A	B	E = 70
30	120	

There are surprisingly many attractive ways to decide how to settle the claims!

Common approaches:

1. Entity equity

Treat all claimants alike no matter what the size of their claims.

(Example of entity equity: US Senate - each state gets two seat regardless of its population.)

What happens when you use entity equity in this example?

A	B	E = 70
30	120	

Answer: Give each player 35.

Many feel it is not "fair" to give a claimant more than they asked for!

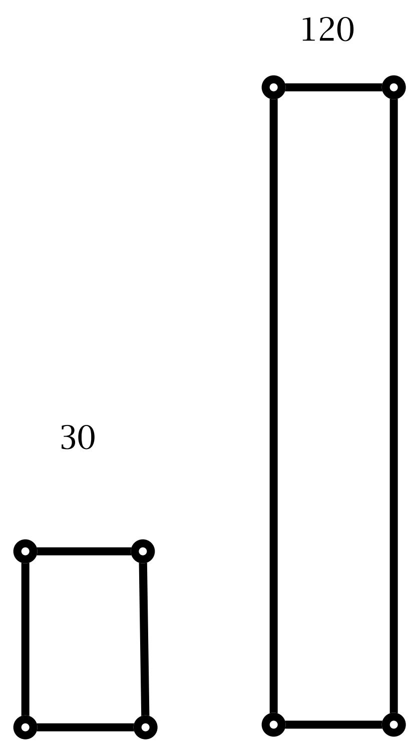
How to modify entity equity to accommodate not giving a claimant more than he/she asked for goes back to Maimonides (1138-1204) - born in Cordoba, Spain; later personal physician to Sultan Saladin.

2. (Maimonides gain) Equalize as much as possible the amount given to each claimant but never give a claimant more than he/she asked for!

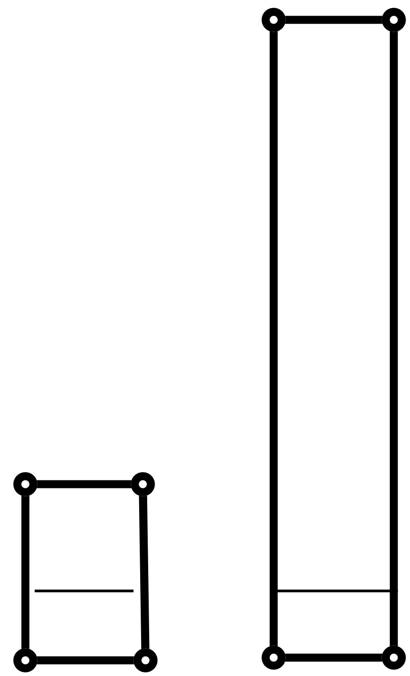
This is what is known as a constrained optimization problem. Can be solved using mathematical programming tools.

One way to think of solving this kind of problem for two or more claimants is to think of the claims as bins and use a pitcher of fluid with the amount of fluid equal to the size of the estate to fill the bins. One can't pour more fluid into a bin than its capacity (the bin would overflow)! When the pitcher runs out of fluid one is done.

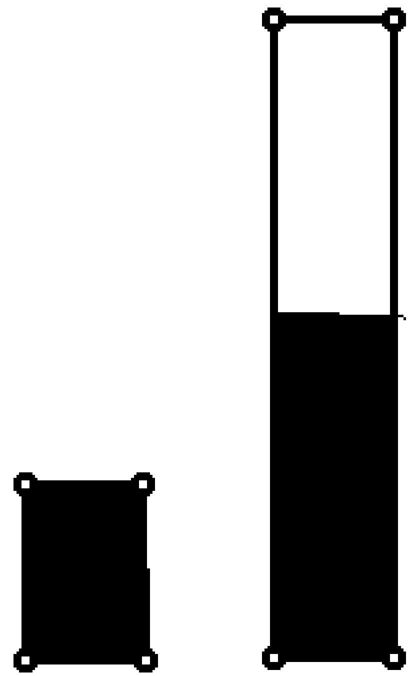
Bins to help solve a claims problem with claims of 30 and 120, E = 70. (diagrams not to scale):



This diagram illustrated how one tries to equalize payments until some claimant(s) gets all of its (their) claim(s) before the estate runs out:



A = 30 B = 120 E=70



30 40

(final amounts to claimants at the bottom)

3. Equalize losses between the claimants even if this means having some claimant subsidize the settlement.

A = 30 B = 120   E = 70

Give: a to A; b to B. a + b = 70 (whole estate distributed); 30-a=120-b
Solve: a + b =70 -a + b = 90 Solution: b= 80, a = -10 (??). A's loss is 40 B's loss is 40!! (since: a = -10; b = 40)

In the approach above equalizing loss to claimants allows one to have a context to teach how to solve systems of linear equations in K-16 mathematics classes.

Another approach:

Since the total loss is 80, so each player should lose 40. We must arrange for the gain to be so that each player's loss is 40. Since

$A = 30 \quad B = 120 \qquad E = 70$

Give $a = -10$ (loss 40) to A and $b = 80$ to B (loss 40). Note: $-10 + 80 = 70$.

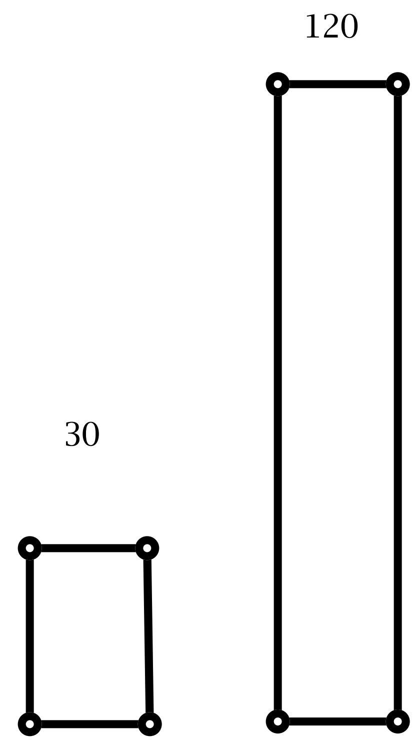
Does not seem fair to many people, because A must subsidize the settlement, and might not have the money to do this; but even if A does it seems not quite right.

Think about: Might there be some contexts of bankrupcty problems where this makes sense?

4. Maimonides loss

Equalize losses as much as possible without forcing any claimant to subsidize the settlement.

Can use the bin diagrams but pay down from the larger claims to try to equalize losses.



Initially, A's loss is 30; B's loss is 120. Start paying down B's loss to equal A's - All 70 goes to B. At this stage B's lose is 50, and A's 30. Without subsidizing

this is best possible! A gets a = 0; B gets b = 70!

5. Proportionality (gain)

Give each claimant the portion of E that arises from its percentage of the total claims.

A = 30 B = 120   E = 70

Total claims: 150; A's share = (30/150)(70) = 14; B's share = (120/150)(70)=56

6. Proportionality of loss:

Total loss is 80. A's share is 1/5 and B's share is 4/5. Hence, A's loss should be 16; B's loss should be 64. Thus, A gets 14 and B gets 56.

Same as proportional solution and always is! This is a general theorem.

Note: If one pays a fixed amount on the $ to the claimants, what do they get? Let x be this amount on the $

$30x + 120x = 70$
$150x = 70$

$x = 7/15$ so $(7/15)(30) = 14$ for A and $(7/15)120 = 56$ for B.

Check: $14 + 56 = 70$ (same as proportional solution.)

Another approach to proportionality is to allow the estate E to grow to the total size of the claims at an interest rate of i percent a year by time T. Now compute the present value of the amounts that would be paid at time T. Once can prove this approach yields the proportional solution.

7. Concede and divide (sometimes called the contested garment rule and sometimes the Talmudic Method).

Not easy to "invent" but seems very appealing. Only applies to case of 2 claimants.

$A = 30 \quad B = 120 \qquad E = 70$

B says to the judge: A is asking for only 30 so of the 70, 40 must be mine! B's uncontested claim against A is 40. A has no such claim against B. Give each player their uncontested claim and split the remainder equally:

A gets $0 + 15 = 15$; B gets $40 + 15 = 55$

Example 2:

A	B	E = 90
40	80	

B says to the person distributing the estate: A is asking for 40 of 90, so 50 of the money must be mine! This is B's uncontested claim against A.

A	B	E = 90
40	80	

A says to the person distributing the estate: B is asking for 80 of 90, so 10 of the money must be mine! This is B's uncontested claim against A.

So the uncontested claims add to 60 leaving 30 to split equally:
A = 10 + 30/2 = 25; B gets 50+15=65

Can something like this be done with 3 or more claimants?

Typical example:

Claims: E is available to distribute

A = 30 B = 90 C = 120 ; E = 150

The Talmudic method (reviewed below) is *consistent*. When applied to 2 claimants it gives the same result as concede and divide.

Other methods are consistent as well.

There are many papers concerned with exactly what axioms a method to solve a bankruptcy problem must hold in order for one to be *forced* to use a particular method.

Axiomatics is important in social choice as well as geometry and algebra.

The "definitive" reference here is:

William Thomson, How to Divide When There Isn't Enough (Subtitle: From Aristotle, the Talmud, and Maimonides to the Axiomatics of Resource Allocation)

Cambridge U. Press, 2019, (481 pages)

Revival of this topic is due to Maschler and Aumann (1985):

Example: (Review)

A = 30 B = 90 C = 120   E = 150

Half claims: A = 15, B =45, C= 60

So these can be met with Maimonides gain:

A = 15, B = 45, C = 60 (120 used)

Since E was 150 we have 30 units left to settle the other half-claims using Maimonides loss:

Half claims: A = 15, B =45, C= 60

Give C 15 units to reduce her loss to 45. Now we can give B and C each 7.5 to try to equalize losses as much as possible.

So A: $15 + 0 = 15$

B: $45 + 7.5 = 52.5$

C: $60 + 15 + 7.\ 5 = 82.5$

These add to $E = 150$

A partial test of "consistency."

Check B and C. Their claims were 90 and 120. How would the these claims be settled with the 135 units they were given, using the Talmudic method? Half claims are B = 45 and C = 60; settle these with 105 units, leaving 30 left.

Now settle: B = 45 and C = 60 based on losses. Give C, 15 to reduce C's loss to B's of 45, and split the remaining 15 units between them - 7.5 to each.

B's share is 45 + 7.5 = 52. 5

C's share is 60 + 15 + 7.5 = 82.5

So we settled B and C's claims in a consistent manner!!

(Same as concede and divide.)

Reference:

R. Aumann and M. Maschler, 1985, "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud", Journal of Economic Theory 36: 195-213

Note: Using first Maimonides loss on 1/2 claims and then Maimonides gain on 1/2 claims is NOT the same method at the Talmudic method!!!

Shapley (Initial problem: $A = 30$ $B = 60$ $C = 90$; $E = 120$

There are 6 permutations to read from left to right:

Shapley (Initial problem: A = 30 B = 60 C = 90; E = 120

ABC so A gets 30, B gets 60 and C gets 30 (30 left after A and B get their 90)

ACB so A gets 30, C gets 90 and B gets 0

BAC so B gets 60, A gets 30 and C gets 30

BCA so B gets 60, C gets 60 and A gets 0

CAB so C gets 90, A gets 30 and B gets 0

CBA so C gets 90, B get 30 and A get 0

A's final allotment is (1/6)(30+30+30+0+30+0) = 120/6 = 20

B's final allotment is (1/6)(60+0+60+60+0+30) = 210/6 =35

C's final allotment is (1/6)(30+90+30+60+90+90) = 390/6 = 65

Check: 20+35+65=120 as required.

We saw many ways to decide an election in an appealing manner; many ways to solve a bankruptcy problem appealingly; many ways to solve an apportionment problem in an appealing manner. However, picking which appealing approach is not so clear, and in may cases there are short lists of fairness properties so that none of these appealing methods obey ALL of the items on a short list of fairness properties!

Mathematics contributed to social choice and fairness problems:

(Kenneth Arrow's Theorem (elections and rankings); Satterthwaite-Gibbard (elections and rankings); Balinski-Young (apportionment); impossibility results due to William Thomson for bankruptcy problems.)

Coalition games:

In bankruptcy problems the players (claimants) are acting on their own for a share of the "pie."

In coalitional games, single or groups of claimants want:

a. To lower their costs

b. To increase their profits

c. Improve outcomes by cooperation

When will individual players cooperate with other players in order to do better?

We can give "normative" advice and also see what actually happens in actual coalition games (European union); lab experiments: behavioral game theory.

Profit sharing:

Corporations get certain amounts separately, but MORE, when they merge.

v({ }) = 0 (the set of no players - the empty set - can't get anything of value!)

v({A}) = 80; v{B}) = 100; v({C}) = 120

v({A,B}) = 184; v({A,C}) = 248; v({B,C}) = 250

v({A,B,C}) = 390

What happens when one shares costs rather than shares profits?

Now coalitions like to see their cost decrease with size!

Thus, a water purification plant system has stand alone costs but costs go down (sometimes) when large units get together to do water purification.

$c(\{\ \}) = 0$ (the set of no players - the empty set - can't get anything of value!)

$c(\{A\}) = 80$; $c(\{B\}) = 100$; $c(\{C\}) = 120$

$c(\{A,B\}) = 150$; $c(\{A,C\}) = 180$; $c(\{B,C\}) = 210$

$c(\{A,B,C\}) = 270$
(less than summing costs to individual players)

This approach to representing a game, when there are many players, is often called the *characteristic function form*, and goes back to Von Neumann. The goal here is to try to predict and/or offer advice to players of such games as to what coalitions it makes sense to form. Ideally, in many cases, the grand coalition which consists of all of the players will arise - everyone cooperates.

If coalitions form, the major question is how the members of the coalition should share the benefit of cooperating.

One is concerned with treating the players who join coalitions fairly (in terms of their "benefits," but one also wants the coalitions to be stable - not have cycles where arrangements are broken to get better arrangements, which return to the start arrangement in a cycle.

There usually are MANY ways to share the amount a coalition gets by acting together. Suppose the "grand coalition" forms - how might one divide the cost this coalition is required to pay out to its members?

Let x_i = amount given to player i in a coalitional game; N = set of all players

Rationality or fairness conditions:

a. $x_i \leq c(\{i\})$ (player i won't join a coalition unless its costs are less when part of this coalition than acting alone!

Sometimes called individual rationality.

b. The sum of the x_i values for members of the grand coalition is the same as c (N).

Sometimes called group rationality. Similar to Pareto Optimality - want the "pie" that is split to be as large as possible.

This approach is especially important when there are many players in the game.

Intuitively, if there are n-players, each player can be a member of a coalition or not, so there are 2^n different coalitions we have to consider, including the { } the empty set, and set of all players, the grand coalition.

For large corporations with many "subdivisions" one must have each division of the corporation charged for various services - accounting, telephone, utilities, photocopy services, administrative assistants, etc. Some of these coalition games find applications in this kind of "cost allocation."

Consider the cost allocation version:

What fairness conditions will hold?

In games of this kind (cost sharing; profit sharing) if the game has a non-empty CORE, then in principle the grand coalition makes sense because there are allocations to the individuals of the grand coalition that do NOT encourage them to leave the grand coalition to improve their outcomes.

Coalition games can have no core (empty core), many points in the core, or a unique point in the core. The formal definition of CORE is a bit complex but the intuitive idea is that these are ways of sharing the benefits from the grand coalition in a way that does not encourage players to break up into smaller groups, or act alone. However, mathematics does not offer specific advice as to how to do the division - pick out one of many CORE points when the CORE is not empty.

Remaining important fairness models:

a. Two-sided markets

Typical application: pairing medical school graduates with hospital residency positions

b. One-sided markets

Typical application: assigning students rooms in a college dormitory

Have a good week!

Questions: email me at:

jmalkevitch@york.cuny.edu

and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html