Notes for Remote Presentation 2:

Game Theory/Fairness Modeling

January 25, 2021

Let us take a moment to see how a geometric tool can help with insight into game theory. It will be useful to you in many other contexts as well.

This topic belongs to the areas of mathematics known as discrete mathematics, combinatorics, discrete geometry and graph theory.

Dots and lines diagrams known as:

*graphs

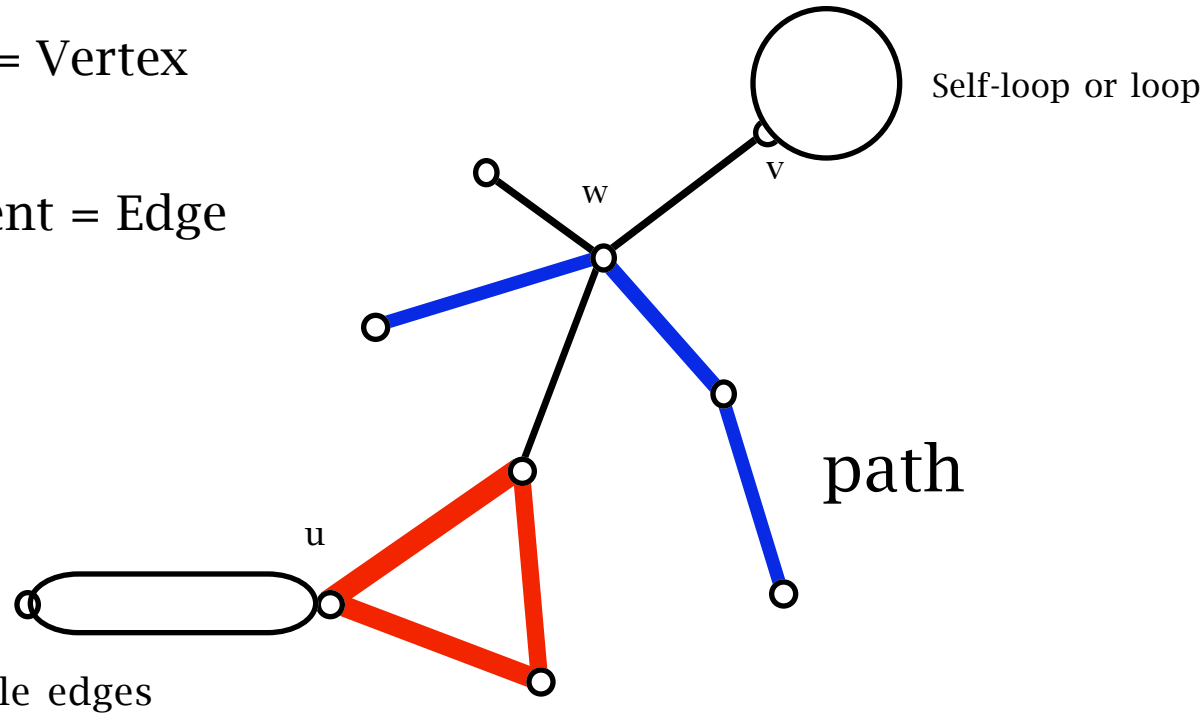
* digraphs (directed graphs) (arrows on the lines)

Introduction (primer) of graph theory:

(Think about how the difficulty of these ideas compare with solving equations! One "downside," lots of new words at the start.)

Dot = Vertex

Line segment = Edge

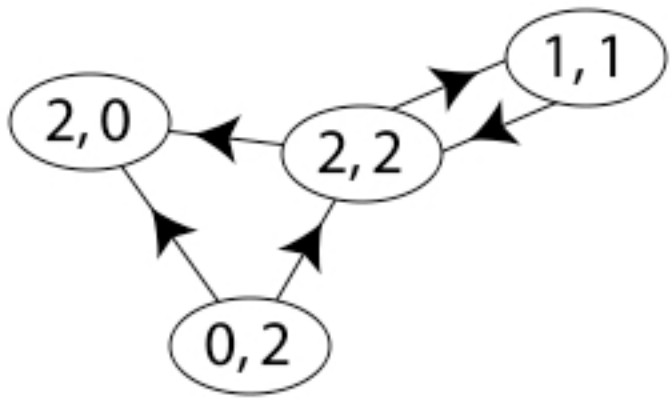


circuit (also called a cycle)

This graph has 10 vertices and 12 edges.

The valence or degree of a vertex in a graph is the number of (local) line segments which meet at the vertex. The valence of v is 3, of w is 5, and of u is 4.

Digraph: 4 vertices; 5 directed edges.

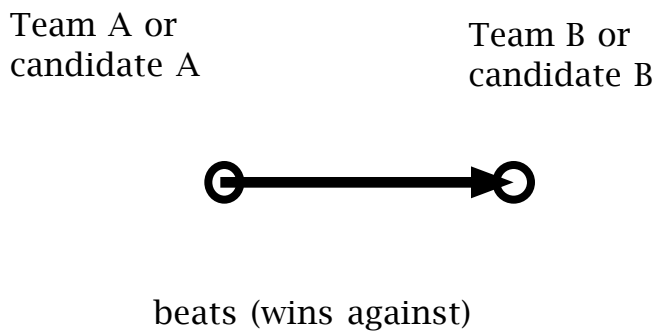


Numbers show indegree and outdegree of vertices.

Indegree: number of directed edges coming into a vertex

Outdegree: number of directed edges leaving a vertex

A major application of digraphs is to the analysis of the outcomes of sports and other tournaments. If team A beats B in a match (or in an election) one draws an arrow from A to B, and often one labels the arrow with the score. (A beats B by 7 to 3.)



Now let us return to thinking about how to play zero-sum games wisely.

Example of a zero-sum game with two actions for each of the two players, Row and Column:

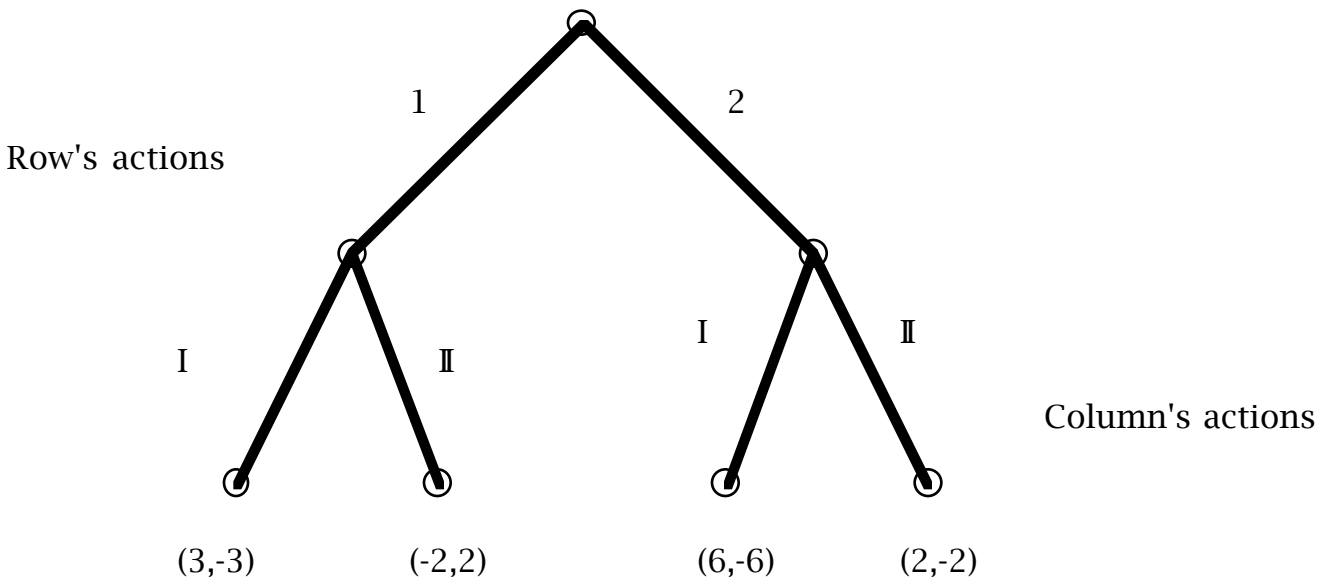
Moves for player named Column

Moves for player named Row.

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(6, -6)	(2, -2)

A more flexible way to represent games than matrices is to use tree diagrams.

Tree of a matrix game:



Outcomes at the leaves are payoffs, Row's is the first number; Column's is the second number.

How might Row and Column reason about how to play such a game?

Moves for player named Column

Moves for player named Row.

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(6, -6)	(2, -2)

For Row: whatever Column does, row 2 is better than row 1! (6>3; 2>-2)

Moves for player named Column

Moves for player named Row.

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(6, -6)	(2, -2)

For Column: whatever Row does, column II is better than column 1! (2>-3; -2>-6)

So it makes sense, for no matter how many independent plays are made of this game, for Row to always play row 2 and Column to always play column II. Payoff every time is: Row wins 2, Column loses 2. The game is UNFAIR: Row always wins, Column always loses when both play OPTIMALLY!

If the players of the original game are playing "rationally," it is as if they were playing the following 1x1 matrix game:

	Column II
Row 2	(2, -2)

A dull game to play especially for Column, who always loses.

Note that Row moves by picking a row to "play." Column moves by picking a column to play.

When one finds a row that dominates another row one can get a SMALLER game matrix by CROSSING out the row which is DOMINATED, leaving the dominating row intact.

So a first step in analyzing how to play a zero-sum matrix game is by looking for rows or columns that might dominant other rows or columns.

Note: Initially there may be NO dominating row but after eliminating a dominated column, there may be a dominating row.

Initially there may be NO dominating column but after eliminating a dominated row, there may be a dominating column.

Thus, Row (the row player) looks for dominating rows.

Thus, Column (the column player) looks for dominating columns.

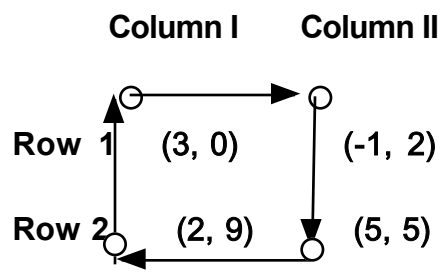
Simplify this **zero-sum** game matrix as much as possible. Payoffs are from Row's point of view. A payoff of -4 is a GAIN for Column and a loss for Row.

	I	II	III
1	2	0	-9
2	3	1	-7
3	1	-4	2

What does one do if dominating strategy analysis does not simplify the game matrix of a zero-sum game?

Here is a tool to help one think about such games better, and uses a digraph.

Example: Motion diagram of a 2x2 matrix game - in this case the payoffs are *not* zero-sum:



One dot for each payoff.

This diagram shows that there is no outcome that is STABLE because one of the players has an incentive to change his/her actions. A "stable" outcome would be one with OUTDEGREE zero.

This game has no dominating rows or columns: How would you play this game zero-sum?

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

Suppose you were required to play this game 100 times - a new round after each prior round is completed.

How would you decide what sequence of moves to make?

Suppose you are Column and you notice that Row always plays this pattern of rows:

1, 1, 2, 1, 1, 2, 1, 1, 2, ...

What would you do?

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

Row's pattern:
Rows: 1, 1, 2, 1, 1, 2, ...

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

If Row said I plan to play Row one for sure, what would Column do?

Column can be sure of getting a positive payoff by playing column II!

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

If Row says I plan to play Row 2 for sure, then Column can be sure to win by playing column I.

Thus, if either player detects a pattern in the play of their opponent, they can exploit that information to get BETTER outcomes!

Thus, for this game:

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

optimal play for both players requires they use of a randomization device!!!

Fact:

For many zero-sum games that are played repeatedly, the best way to play is NOT deterministic but using some way to make one's choices randomly!

But there are infinitely many ways of playing randomly. Is there one which is best?

Answer: Yes! We will see how to find that BEST way of playing randomly.

Review of randomization and probability theory!

When one does an "experiment" there is a collection of possible outcomes, assumed to be finite for our purposes.

Example: Toss a coin:

Outcomes: head or tail

Example: Roll a standard die (plural is dice)

Outcome: 1 to 6 "spots"

Some coins are "fair" and some are "biased."

Some dice are "fair" and some are biased.

A fair coin will show heads and tails in approximately equal frequencies. Thus if one tosses a FAIR coin 20 times, one will get approximately 10 heads and approximately 10 tails, the sum being 20.

However, and this is a sticking point for most beginners, when one tosses a fair coin 20 times the chance of getting exactly 10 heads and 10 tails is very low!!

In fact the chance of this happening is:

.00009765625

About one in ten thousand experiments!

We need to understand the difference between the probability of something happening, and the EXPECTED value of an event.

Expected value is the probabilistic analogue of finding the mean of a collection of numbers:

Mean of: 1, 3, 2, 3, 5

$(1+3+2+3+5)/5 = 14/5 = 2.8$

Note: the mean is not one of the ORIGINAL data values!

A ***fair*** coin is tossed three times. What is the expected number of heads?

Outcomes:	Number heads:
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2
HHT	2
HHH	3

Mean: Sum/8 = 12/8 =1.5

Note: The mean values is not a possible outcome which can only be 0, 1, 2, or 3.

For the general situation where there are two outcomes: A and A' (not A, the complement of set A), from an experiment repeated n times.

The Binomial Model

The expected number of times A occurs will be:

$$p(A)n$$

The number of times A' occurs is: $p(A')\ n$

Example: a fair coin is tossed 21 times:

expected number of heads is:

$(1/2)x21 = 10.5$

Theorem: If we have numerical outcomes (random variable) x(i) where p(x(i)) is the probability of x(i) occurring the expected value is given by: Sum over all outcomes of:

sum: x(i)p(x(i))

For the biased coin $p(H)=1/4$ and $p(T) = 3/4$

$n \times p(H)$ gives: $3(1/4) = 3/4$

$n \times p(T)$ gives: $3(3/4) = 9/4$

Note $3/4 + 9/14 = 12/4 = 3$

Details of the calculation: (sum product of outcome and its probability)

TTT	0	27/64
TTH	1	9/64
THT	1	9/64
THH	2	3/64
HTT	1	9/64
HTH	2	3/64
HHT	2	3/64
HHH	3	1/64

Expected value: $48/64 = 3/4$

Given the set S of n outcomes of an experiment, called the sample space of the experiment:

$$S = \{ o_1, o_2, o_3, \ldots, o_n \}$$

Probabilities of events obey:

1. $p(o_j) \geq 0$ (probability of each event is positive or 0.)

2. $p(o_j) \leq 1$ (probability of each event is at most 1.)

3. The sum of the probabilities of all the sample space events adds to 1.

Note: The standard symbol for summation in mathematics is the Greek letter "capital" sigma:

$$\sum$$

Thus:

$$\sum p(o_i) = 1 \text{ (summed from } i = 1 \text{ to } n)$$

Fundamental principle of counting:

If I have 4 shirts and 5 pairs of jeans I can wear 20 different outfits?

When choices are made independently, we multiply the choices to get the number of outcomes:

m choices; n choices

Total number of possibilities: mxn

This is called the multiplication rule.

If a toss a coin followed by rolling a die followed by rolling another die there are:

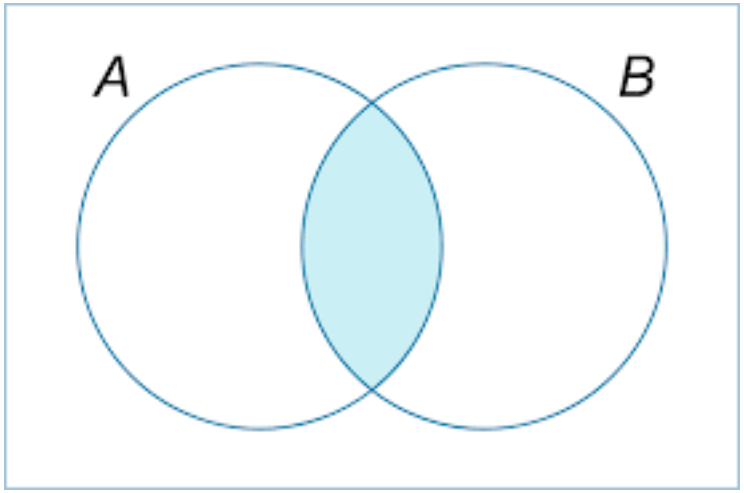
2x6x6 = 72 possible outcomes!

Multiplication rule for probabilities:

Event A: probability p(A)

Event B: probability p(B)

$p(A \text{ and } B) = p(A \cap B) = p(A) \times p(B)$



Outcomes where both A and B occur are shown in green. The outcomes in one or both of the circles is called A union B. $A \cup B$. (A occurs or B occurs or both occur.)

A fair biased coin p(head) = 1/4 is tossed followed by rolling a fair die. What is the probability of getting a tail and a 6?

Since the p(head) = 1/4, p(tail) = 1-1/4

p(tail) = 3/4; p(6) = 1/6

Probability of a tail and a 6 is:

(3/4)(1/6) = 3/24 = 1/8

		1/5	4/5
		Column I	Column II
1/4	Row 1	3	-1
3/4	Row 2	-9	7

p(outcome 3) = (1/4)(1/5) = 1/20.
p(outcome 7) = (3/4)(4/5)=12/20
p(outcome -1) = (1/4)(4/5)=4/20
p(outcome -9) = (3/4)(1/5)=3/20

Note: These numbers account for all outcomes so add to 1.

If the players use these "spinners," what is the payoff to Row?

Row earns:

$(1/20)x3 + (12/20)x7 - 1(4/20) - 9(3/20) =$

$(3 + 84 - 4 - 27)/20 = 56/20 = 2.8$

On each play of the game Row wins on average 2.8 and Column loses 2.8

Can Column do better?

How should one play this game? Is it fair?

	Column I	Column II
Row 1	3	-1
Row 2	-9	7

Optimal play means designing a "spinner" for each player that gives the best outcome for each player, even if it means that one player has an advantage and the game is not fair.

Row plays Row 1, p percent of the time. Column plays Column I, q percent of the time.

		q	1-q
		Column I	Column II
p	Row 1	3	-1
1-p	Row 2	-9	7

Note: (3 + (-1) + (-9) + 7) = 0. **Does this mean the game is fair?**

		q	1-q
		Column I	Column II
p	Row 1	3	-1
1-p	Row 2	-9	7

Value to row: EV =

$$3pq + (-1)(p)(1-q) + (1-p)(q)(-9) + (1-p)(1-q)(7) =$$

$$3pq + pq + 9pq + 7pq - p - 7p - 9q - 7q + 7 =$$

$$20pq - 8p - 16q + 7 = 20p(q - 8/20) - 16q + 7 =$$

		q	1-q
		Column I	Column II
p	Row 1	3	-1
1-p	Row 2	-9	7

$$= 20p(q-8/20) -16q + 7 =$$

$$20p(q-8/20) - 16(q-8/20) +7 - 16(8/20) =$$
$$(20p -16)(q - 8/20) + (140/20) - (128/20) =$$

$$= \quad \mathbf{20(p-4/5)(q-2/5) + 3/5}$$

What have we learned?

EV(Row's viewpoint) = 20(p-4/5)(q-2/5) + 3/5

Row should play Row 1, 4/5 of the time.
Column should play Column I, 2/5 of the time.

The game is not fair: Row win 3/5 on average with every play of the game. Column "gains" -3/5 (loss)

Amazing fact: When Row plays Row 1, 4/5 of the time it does not matter what Column does because the first term above is 0!! (Any number times 0 is 0!) When Column plays Column I, 2/5 of the time it doesn't matter what Row does. Column gets the best outcome possible against a "rational" Row player.

General Theorem: (Solution of the game.)

For a 2x2 game zero-sum game with no stable point (motion diagram is a cycle) optimal play requires using randomization and their payoffs are governed by an expression of the form:

$$EV = C(p-a)((q-b) + K$$

where p and q are the percentage of the time that Row plays Row 1 and q is the percentage of the time that Column plays Column 1. The game is fair when K = 0, otherwise it is not.

Using the theorem to simplify calculations:

If Row want to design an optimal spinner for a 2x2 game with no stable point, Row knows that the payoff gotten will be the same if Column always plays Col I or always plays Column II. We can use this to find the value of p for Row's optimal spinner.

		q	1-q
		Column I	Column II
p	Row 1	3	-1
1-p	Row 2	-9	7

Thus:

3p -9(1-p) = -p + 7(1-p) (*)
12p-9 = -8p + 7
20p=16
p=4/5 (the same value we got from the earlier more complex calculation. EV(row) = 3(4/5)-9(1/5) = 3/5 (or -(4/5) + 7(1/5) = 3/5). Substitute 4/5 in one side of (*))

(This confirms the earlier result).

No dominating row or column:

Row/Column	I	II	III
1	-3	2	-4
2	2	-6	-5
3	-7	3	-6

(Row 1, Col III) is a stable point. Minimum in its row and maximum in its column.

This choice is optimal. "Nash equilibrium."

To get further insight we have to look at games which are not zero-sum:

	Column I	Column II
Row 1	(3,3)	(-4,4)
Row 2	(4,-4)	(-1,-1)

What is rational play in this game? Answer next week!

Mathematical model:

A simplification of a situation in the "real world" so that one can carry out an analysis.

Mathematical models can be physical or "conceptual."

Example of a physical model:

Give examples where voting and elections are used in our lives:

What are the features (components) of an election or voting system that we are trying to understand elections so we can "improve" the way the election is carried out?

Components of an election or voting system:

1. One needs voters or decision makers.

Say n voters or decision makers.

2. One needs alternatives or candidates to vote on or choose from.

Say there are m candidates.

3. One needs a way for the voters to express their opinions about the choices or candidates.

The usual way this is done is by using a ***ballot***. We also need to think about how voters behave in filling out ballets.

4. Based on the ballots one needs a way of deciding who the winner or collection of winners is. Sometimes one is filling seats on a committee and there may be several people elected.

What types of ballots are you familiar with from elections in which you have participated?

In America we vote for:

* President

* Members of the House of Representative and Senate

* Governors

* Mayors

* Chairperson of a department

* Faculty committees

* Best actress

* Best movie

* Best rookie pitcher

* Best player in a particular football game

Mathematics has explored the surprisingly many ways to to construct ballots as inputs to elections.

The the major distinction parallels the two major kinds of numbers we use:

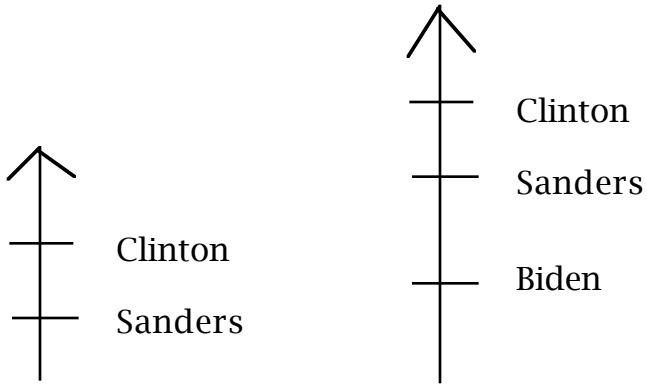
ordinal (counting numbers)

cardinal numbers (to measure)

* ordinal or rank ballots (with or without ties)

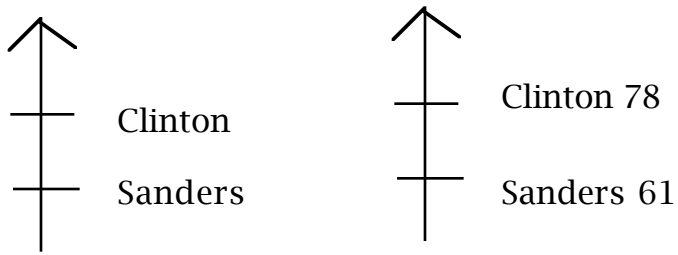
Show order of the candidates (choices) but not how strongly one feels about the candidates.

Two and three candidate ordinal ballots:



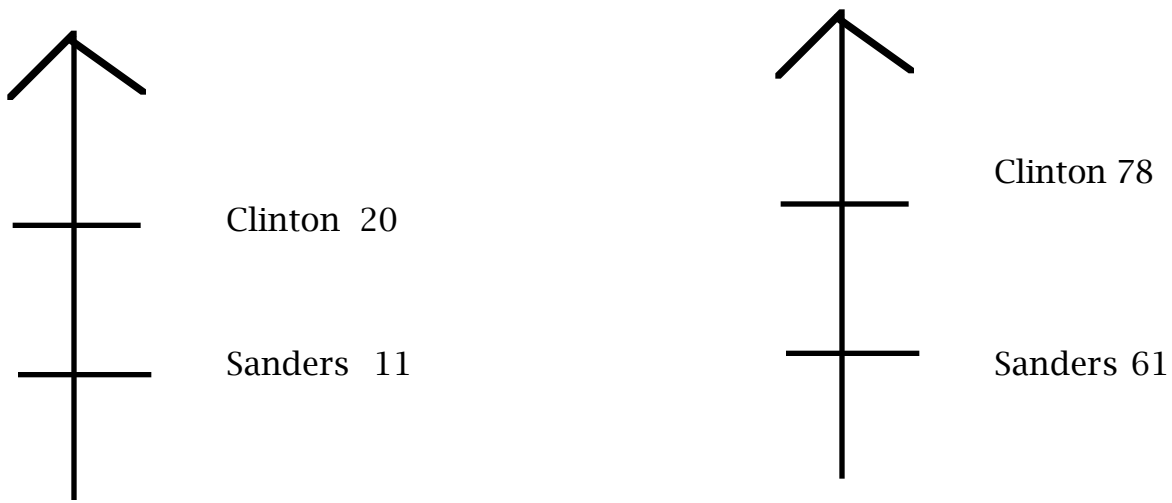
* cardinal ballots show intensity of support

Scale (100 high; zero low)



Same ranking but very different information about "intensity."

Scale: 100 high; 0 low



Is ballet truncation allowed?

Truncation on a ballot refers to the voter not listing all of the candidates but only some of the candidates.

Truncation can occur because the voter chooses to not list candidate he/she knows or the voter may not know anything about the candidate.

Sometimes a ballot is truncated because the voter knows what method is being used to count the ballots, and voting for more than one candidate will help not only one's favorite but other candidates as well. This called ***strategic voting***.

Such voting is called strategic. One "lies" about one's true views about the candidates to help a particular candidate or group of candidates win. Thus, one might only rank "conservative" candidates in a primary election with many people seeking office.

In some elections TRUNCATION is not permitted! Sometimes votes must rank the candidates without ties.

Have an enjoyable week!

If you have questions email me:

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class web page:

https://york.cuny.edu/~malk/gametheory/index.html