Notes for Remote Presentation 6:

Game Theory/Fairness Modeling

February 22, 2021

If you were the Row player how would you play this game? Note: there is a tie for two of Row's payoffs.

	Column I	Column II
Row 1	(5, 5)	(-100, 4)
Row 2	(0, 1)	(0, 0)

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What are the Nash equilibria (if any?) for this game?

	Column I	Column II
Row 1	(5, 5)	(-100, 4)
Row 2	(0, 1)	(0, 0)

We know there must be Nash equilibria?

Row 1, Column I is a Nash equilibrium in pure strategies.

	Column I	Column II
Row 1 p	(5, 5)	(-100, 4)
Row 2 1-p	(0, 1)	(0, 0)

If Row tries to equalize Column's two columns in column's game: $5p+1-p = 4p$, which has no solution.

So there is no mixed strategy Nash equilibrium.

	Column I	Column II
Row 1 p	(5, 5)	(-100, 4)
Row 2 1-p	(0, 1)	(0, 0)

Even is both players "agree" to play Row 1 and Column I, the Column player may be tempted to "punish" Row by playing Column II, getting slightly less, but Row has a big loss!

In this game Column can never lose but if Row loses the loss is quite large compared to the gain for either player.

Behavioral game theory is a branch of behavioral economics and involves experiments which see whether "real world people" actually play games according to the way game theorists argue rational players should play them?

The first woman mathematician to be President of the American Mathematical Society was Julia Robinson, married to another mathematician, Raphael Robinson, (now both deceased). Julia's sister, Constance Bowman Reid, (also deceased) was the famous historian of mathematics, who wrote a biography of her sister and Richard Courant, David Hilbert.

Julia Robinson on the left; Constance Reid on the right.

(Wikipedia)

Julia Robinson proved a theorem suggested by George W. Brown, concerning "learning" in the play a zero-sum matrix game:

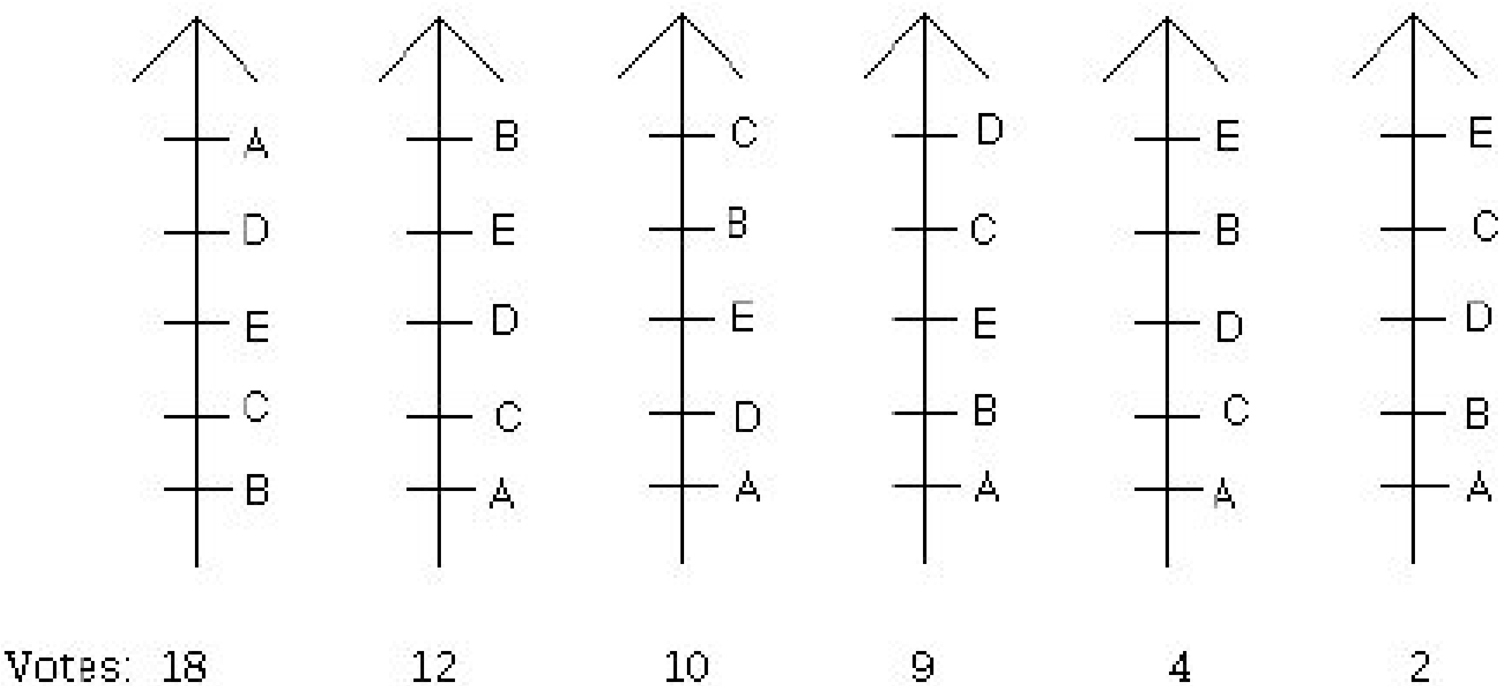
If each player chooses in turn the best pure strategy against the relative frequency mixed strategy of his/her opponent up to then, then the player's strategy converges to the optimal mixed strategy for the game.

(This approach is called "fictitious" play.)

Do experiments confirm that this happens?

Voting and Elections:

Typical (?) Election ballot data:



Who deserves to be the winner of this election?

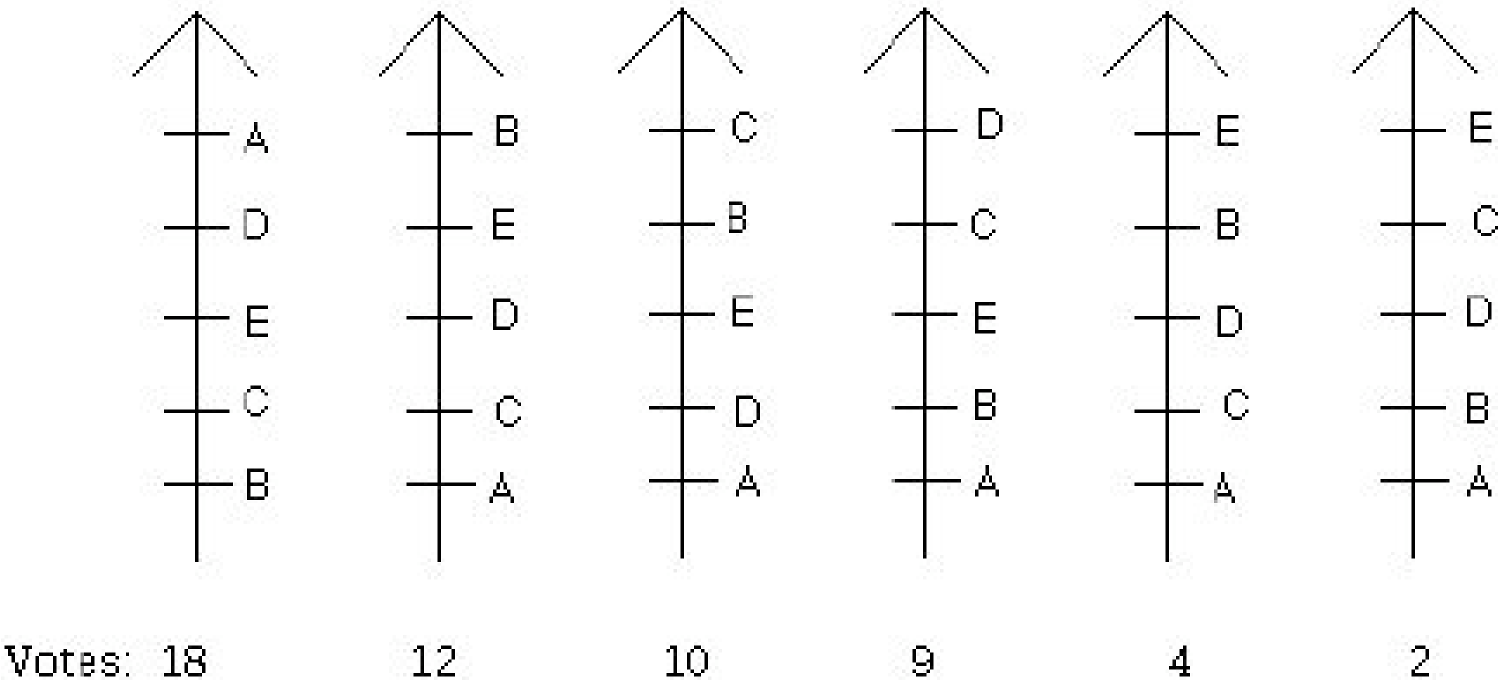
Whoever wins is usually viewed as the "people's choice" and expresses the "will" of the people.

American's tend to have faith that the "best" candidate wins in such important elections as those for President. Is this actually true?

Part of the reason that what happened on January 6 was so shocking was that peaceful transfer of power is one of the bedrock aspects of being a democracy.

Political parties tend to have faith that their "best" candidate wins in a primary election. Perhaps with plurality voting this is not true?

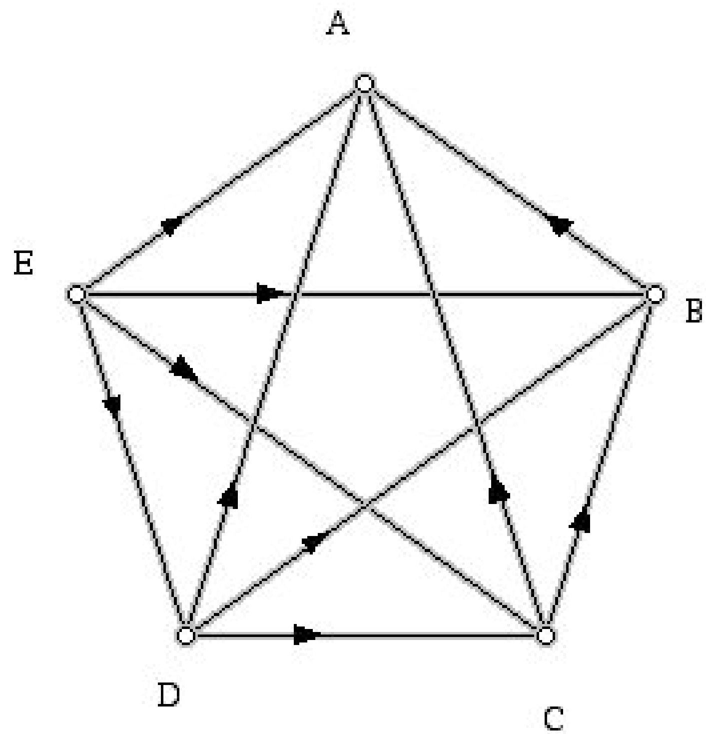
Challenging election to pick a "good" winner:



Using tradition as a guide, the way nearly all American elections are decided, A would be the winner!

Even though a majority of voters liked A the least and A's percentage of first place votes is only about 33 percent A is the plurality winner.

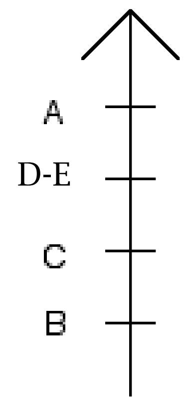
Here are the results of 2 way races: Perhaps you are surprised that E can beat each of the other candidates in a two-way race:



Perhaps the voting system should depend on "average" strength of the candidates with the voters.

Give points for how high up on a ballot the different candidates are.

This system is called the Borda Count, and the number of points for a candidate is the number of candidates below him/her.



A gets 4 points
B gets 0 points
C gets 1 point
D gets two point
E gets two points

If 10 voters with this ballot, multiply by 10.

Nifty and perhaps unexpected way to get Borda Count results when there is no tie.

Construct a matrix showing votes for the alternatives (candidates) in two-way races.

Here is the data about the two-way races

The entry in row i and column j is the number of votes for the row candidate in a race against the column candidate

	A	B	C	D	E
A	-	18	18	18	18
B	37	-	16	26	22
C	37	39	-	12	19
D	37	29	43	-	27
E	37	33	36	28	-

Remarkably: the row sums give a candidate's Borda Count:

B's Borda Count = (0)(18) + 4(12) + 3 (10) + 1(9) + 3(2) + 1(2) = 101

It turns out that D has the highest Borda count so D wins using this election method.

Here are some other methods sometimes used to decide elections using ordinal ballots:

Run-off

If no candidate gets majority that one eliminates all but the two candidates with the highest number of first place votes and conducts an election between them

In our example:

C, D, and E are eliminated and in the two-way race between A and B, B wins.

Sequential run-off

Instead of eliminating lots of candidates with not many first place votes all at once, one can eliminate them one at a time.

Round 1

A = 18; B = 12; C = 10; D = 9; E = 6 (Eliminate E)

Round 2

A = 18; B= 16; C =12; D =9 (Eliminate D)

A = 18; B =16 C= 21 (Eliminate B)

A = 18; C = 37   C wins!

In this election 5 reasonably appealing methods all yield different winners. The winner is not the "best person" for the job but an artifact of the method of counting the votes!

This observation is a very important contribution of mathematics to political science!

USA Presidential Election:

Direct election results:

a. ballot

b. decision method

Electoral college results:

a. weighted voting

b. weights are set by Huntington-Hill algorithm for the apportionment problem

To appear on the a state's ballot for election of the President a candidate must get the nomination of a party (the major ones are the Democratic and Republican parties) or get signed petitions from a large enough collection of voters.

The parties select nominees through a complicated process of primaries and caucuses. However, when a primary occurs, as shown by the recent (2016) Republican primaries, a candidate can "win" a state based on a small plurality of those who vote.

Arrow's Theorem

When there are 3 or more candidates no election method obeys a short list of fairness properties!

Kenneth Arrow majored in mathematics at City College and got his doctorate at Columbia.

Arrow's "fairness" conditions:

Voters produce ranked ballots with ties. One seeks a ranked ballot with ties for "society."

a. Procedure works for any ballots.
b. Non-dictatorial
c. Non-imposed
d. Monotone
e. (IIA) Independence of irrelevant choices

a. (Universal) No set of ballots is considered too "strange" to be considered for counting. The elections decision "committee" will not "edit/censure" a set of ballots.

b. No matter the election ballots society does not always agree with how a particular voter voted (no dictator).

c. Not imposed. If all voters vote for a particular candidate in first place that candidate should win.

d. Monotone - more support can't hurt; less support should not make one do better.

e. Independence of Irrelevant alternatives (IIA).

Relative position of X and Y should depend only on information about X and Y.

IIA has typically generated the most skepticism of the Arrow's axioms as regards its importance.

However, it is also tied up with with views about the interpersonal comparison of utility.

Gibbard-Satterthwaite Theorem

When there are three or more candidates all election methods except dictatorship can be "manipulated!"

Gibbard is a philosopher and Satterthwaite a management scientist.

The Borda Count seems like an appealing method though it does not obey Independence of Irrelevant Alternatives.

One obvious criticism: Suppose a voter knows the election will be close between his/her first and second place choices in an election when the voter provides honest preferences. By being honest, the voter's second place choice may "harm" his/her first place choice.

So a voter might be tempted to not vote honestly, but put his/her second place choice towards the bottom of his/her ballot to cut down on that candidate's points. However, all voters may vote dishonestly resulting in an outcome not at all in the interests of the group, or representative of their true opinions.

Weighted voting:

One person, one vote is an important notion in election theory for the United States. (Except the electoral college does not work that way!) More educated or wealthy Americans still only get one vote for who should be President.

How should one have representation in an amalgamation of countries of different sizes such the European Union or the counties of upstate New York where there are towns of very different sizes in a typical county?

County governments in New York provide funds for roads, county police, preventing river/lake contamination, etc.

Sometimes to be fair, different players should have different influence so voting is done using the idea of weighted voting.

Some players cast a bigger vote than others. Thus, each player i will cast a block of votes.

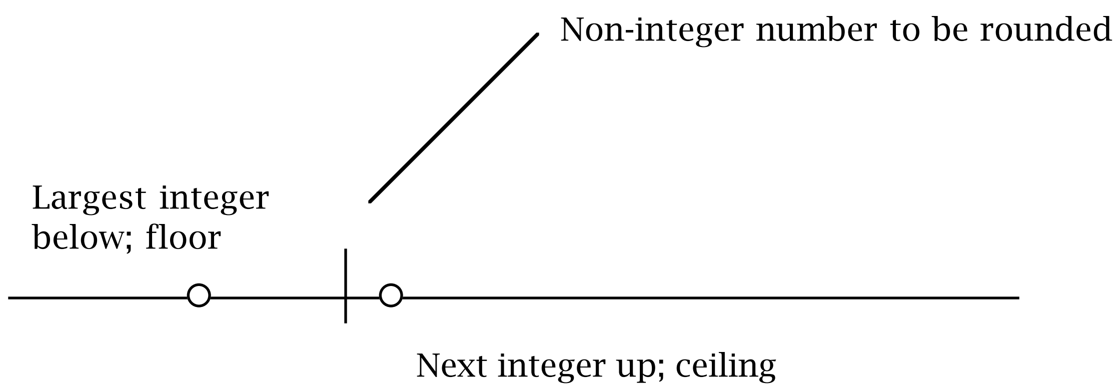
This leads to the idea of weighted voting. One has players 1, 2, 3,..., n each of whom casts a block of votes.

So player i casts w(i) votes, called i's weight.

A collection of players is called a *coalition*. To take action one needs to have sum of the weights of the players in a coalition exceed some number Q, called the Quota. Often the quota is set at the integer above 1/2(sum of weights) + 1

However, in some cases more than a simple majority is needed for action.

Rounding: Floor and ceiling functions: (Notation due to Donald Knuth, a computer scientist.)



$$\lceil 3.12 \rceil = 4 \text{ (ceiling)}$$

$$\lfloor 5.88 \rfloor = 5 \text{ (floor)}$$

Notation: players are "named" so that the small numbered players cast the largest votes (ties allowed).

Example:

[8; 6, 5, 2, 1]

Four players: player 1 casts 6 votes, player 2 casts 5 votes, player 3 casts 2 votes, and player 1 casts 1 vote.

[8; 6, 5, 2, 1]

For an action to be taken a group of "players" with at least weight 8 have to "work together." Thus, {1,2} is a coalition with players 1 and 2 and since 11 is bigger than 8, they can make sure that a "bill" passes the legislature. {1,2) is called a winning coalition.

[8; 6, 5, 2, 1]

{3,4} command only 3 points and so by themselves can't take action. This is a losing coalition.

A player is called a *veto* player if that player is a member of every *minimal winning* coalition.

That is, a group of players who if any player is deleted from that coalition can no longer take action. (Win)

Question: County Z has 5 towns with populations of 900,000, 500,000, 500,000, 400000, and 200,000. What might be a good set of weights and a quota for a county legislature with 5 players?

Weighted voting game:

$$[\ 13;\ 9,\ 5,\ 5,\ 4,\ 2]$$

Minimal winning coalitions:

{1, 2} {1, 3} {1, 4} {2, 3, 4}

Player 5 has NO power! Player 5 is never a member of any MINIMAL winning coalition. That is, a group of players who if any player is deleted from that coalition can no longer take action. (Win)

Example:

[5; 4, 3, 2] Three players named 1, 2, and 3 who cast 4, 3, and 2 votes respectively. The 5 is called the quota. Players with combined weight of 5 are needed to take action.

Is Player 1 twice as powerful as Player 3 because 4 is twice 2?

$[5; 4, 3, 2]$

Which coalitions (collections) of players can take action?

Minimal winning coalitions - no subset of a minimal winning (MW) coalition wins:

$\{1,2\}, \{1,3\}, \{2,3\}$

Given [5; 4, 3, 2], we have total symmetry here for the MW. The MW coalitions are:

{1,2}, {1,3}, {2,3}

so it should be apparent that in this game all three players have equal influence!!!

An isomorphic game would be:

[2; 1, 1, 1]

because its minimal winning coalitions are also:

$$\{1,2\}, \{1,3\}, \{2,3\}$$

Power indices: (Variants differ in using all winning versus MW coalitions) (Name are not standardized.)

a. Coleman

b. Banzhaf

c. Shapely

d. Deegan-Packel-Johnston

[5; 4, 3, 2]

MW: {1,2}, {1,3}, {2,3}

Coleman:

1 is in two coalitions
2 is in two coalitions
3 is in two coalitions

So 1 has 2/6 as a power; 2 has 2/6 as a power; 3 has 2/6 as a power!

Look at the pattern of Yes and No votes of the 3 players:

YYY wins
YYN wins
YNY wins
YNN loses
NYY wins
NYN loses
NNY loses
NNN loses

Underlines show when a Yes changed to a No changes a win to a loss. So of the underlined items each player has 2 out of a total of 6. (This is Banzhaf Power.)

So each player has equal Banzhaf power.

Note: We only look for "pivots/swing," that is changes when a sequence of Y's and N's wins, and changing a Y to an N makes a win a loss.

It turns out that looking at situations where a pattern yields a loss and changing a No to a Yes wins, just doubles the number of pivots/swings because we are computing a ratio.

Shapley-Shubik Power Index

[5; 4, 3, 2]

1 *2* 3
1 *3* 2
2 *1* 3
2 *3* 1
3 *1* 2
3 *2* 1

Pivot player is shown in italics - second in every case for this example.

Hence:

Player 1 has 2 pivots out of 6; power 1/3

Player 2 has 2 pivots out of 6; power 1/3

Player 3 has 2 pivots out of 6; power 1/3

Remember that 2/6 is the same fraction as 1/3.

Banzhaf was not trained as a mathematician. He was trained in the law. He is most famous for winning cases against tobacco companies that smoking is harmful to one's health.

He also won a Supreme Court decision which overturned the use of weighted voting in Nassau County because there were players with NO Power!

Nassau and Suffolk now have legislatures rather than weighted voting but most upstate NY Counties have weighted voting procedures.

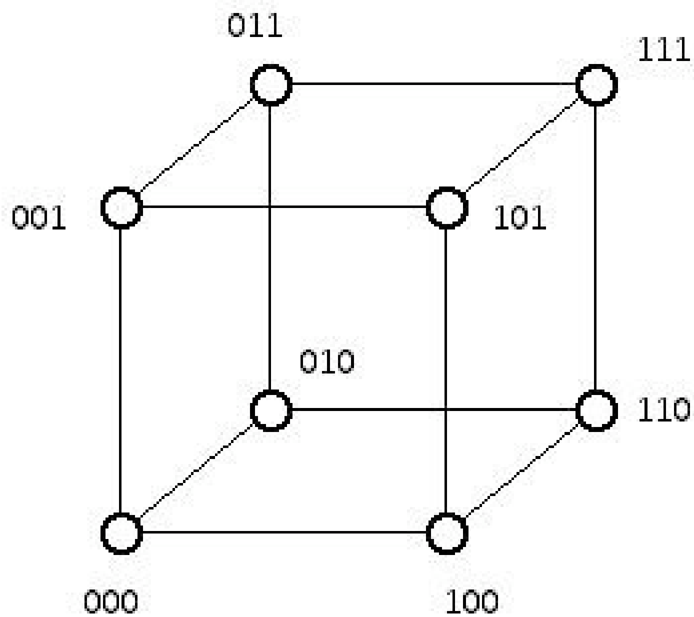
In NYS weights must be assigned to the players in the weighted voting games for county governments so that the Banzhaf Power is proportional to the population of the players involved.

Pattern of Yes/No for lines in a Banzhaf power table for 3-players "corresponds" to the labels needed for a 3-dimensional cube:

NNY,YNY,NYY,YYY(top)

NNN,YNN,NYN,YYN(bottom)

Think of N as a 0 and Y as a 1:



Three-cube made from two 2-cubes! Top layer all entries end in 1; bottom layer all entries end in 0!

Banzhaf table for 4 players correspond is obtained by pasting together two copies of a 3-cube to get a combinatorial 4-cube.

Have a good week!

Questions: email me at:

jmalkevitch@york.cuny.edu

and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html