Notes for Remote Presentation 9:

Game Theory/Fairness Modeling

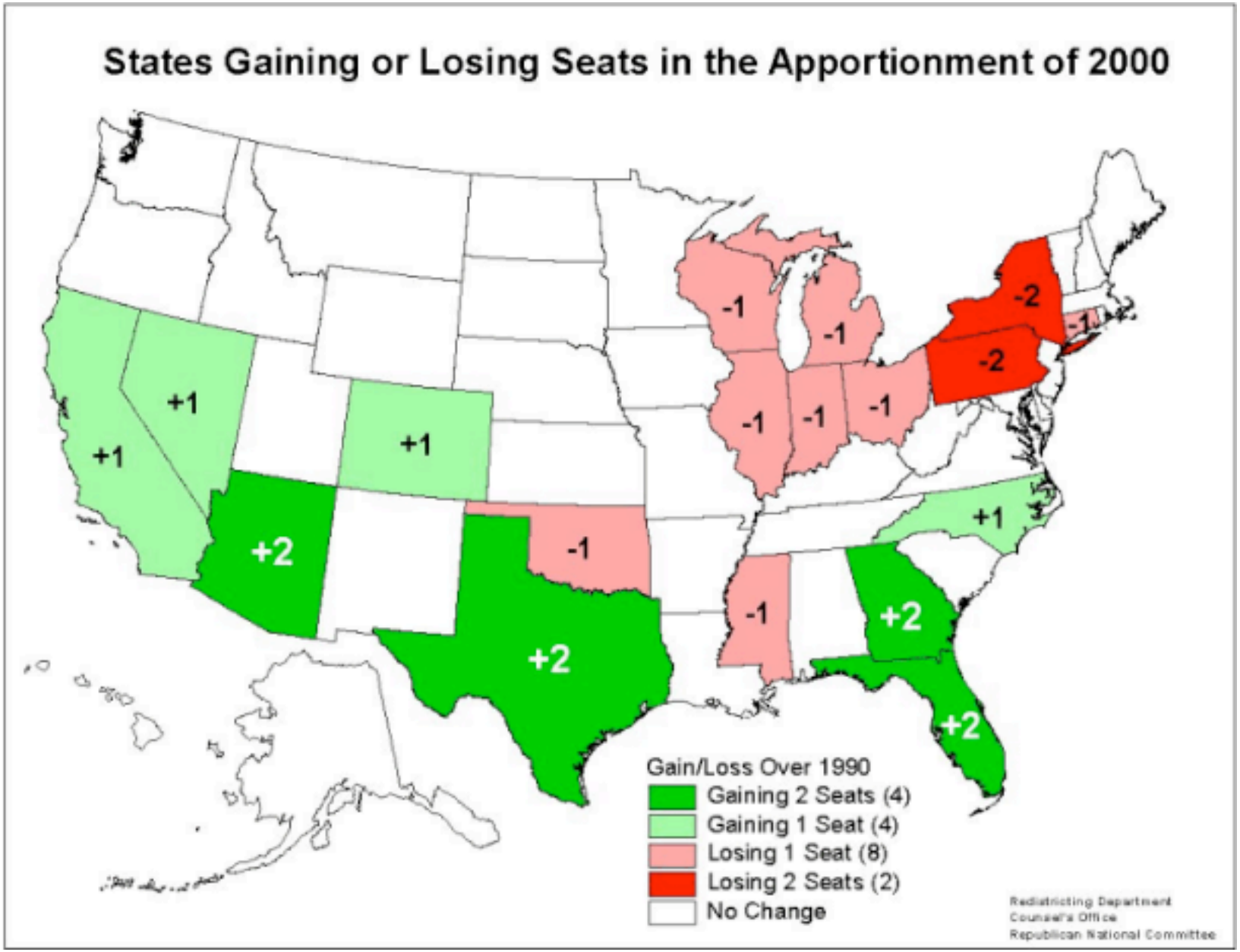
March 22, 2021

Apportionment problems are concerned with assigning a positive integer number of items, with a positive integer number h items available to assign, to claimants based on the size of their claims.

Examples:

a. Assigning seats to parties in a parliament based on the percentage of votes that different parties received.

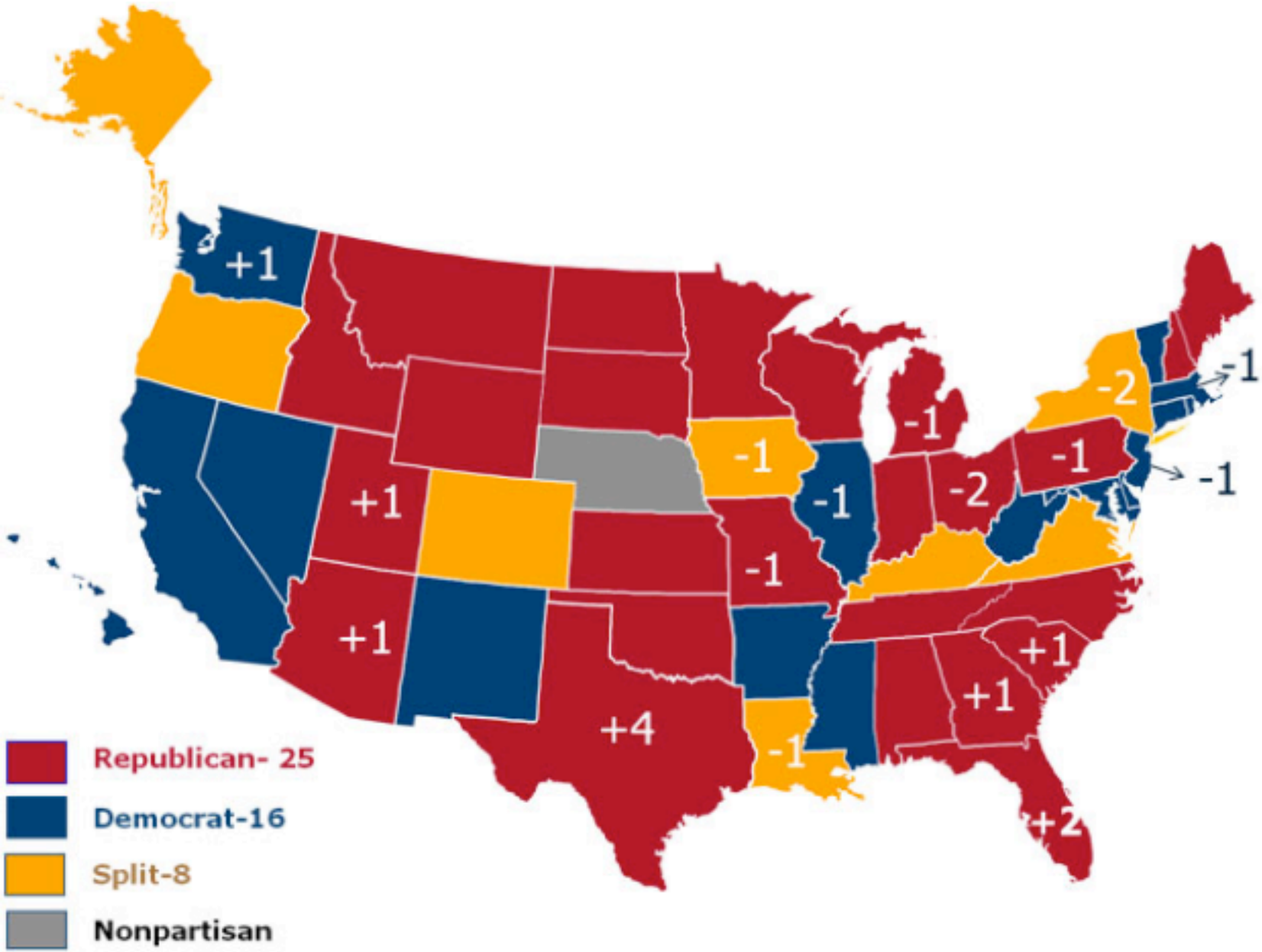
b. Assigning seats to states in the US House of representatives based on the percentage of populations that different states have based on the US Census held every ten years.


States Gaining or Losing Seats in the Apportionment of 2000
+1
+1
+1
+2
-1
+2
-1
-1
-1
-1
-1
-1
-2
-2
-1
+1
+2
+2
Gain/Loss Over 1990
Gaining 2 Seats (4)
Gaining 1 Seat (4)
Losing 1 Seat (8)
Losing 2 Seats (2)
No Change
Redistricting Department
Counsel's Office
Republican National Committee

Similar map for 2010 census:

States *gaining* one or more seats in Congress: Arizona, Florida (+2), Georgia, Nevada, South Carolina, Texas(+4), Utah and Washington.

The states *losing* one or more seats in Congress: Illinois, Iowa, Louisiana, Massachusetts, Michigan, Missouri, New Jersey, New York (-2), Ohio (-2) and Pennsylvania.


+1
+1
+1
+4
-1
-1
-1
-1
-2
-2
-1
-1
-1
+1
+1
+2
Republican- 25
Democrat-16
Split-8
Nonpartisan

Comment: Similar map for 2020 Census has been delayed by court cases initiated during the prior presidential administration.

c. An alumna of a NYC high school has donated 46 lender tablet computers to her alma mater. How should these items be made available to the freshman, sophomore, juniors and seniors at the school based of numbers of students at each of these levels.

d. Assigning a consignment of h respirators to the hospitals in a given city during an epidemic based on number of patients they treated for the ailment in the recent past (or other criterion).

We have looked at one "divisor method" based on the usual rounding rule - Webster/Saint Lague.

What happens if we "round" differently?

Example:

h = 12; Total "population" = 1000

Claims:	A	B	C
	480	320	200

Example:

Claims:	A	B	C
	480	320	200

Total population: 1000

Apply D'Hondt's divisor method based on rounding "fair share" down (floor function).

Exact fair share:

1000/12 = 83.3333.....

Since the there are 1000 people and 12 seats, 1000/12 means each claimant should get seats as follows:

A should get 480/83.33=5.76
B should get 320/83.33 = 3.84
C should get 200/83.33= 2.40
Round down: A=5; B = 3; C=2

Too, few, seats.

Use adjusted quota of 79 instead of 83.33! (Not far off.)

A should get 480/79=6.08
B should get 320/79 = 4.05
C should get 200/79=2.53
Round down: A=6; B = 4; C=2

Assigns 12 seats as required.

Note that though we did not "force" each claimant initially to get one seat, each claimant did get one seat. So this is the same apportionment as if we had used Jefferson's method, which assigns one seat at the start to all claimants.

Jefferson's method differs from D'Hondt in making sure that each state always gets one seat, which for some "data" will not happen unless it is "forced" to happen.

How does this compare with applying Hamilton's and Webster's methods to the same data?

You can try this for practice.

So far we have not discussed what to do in the case there are ties. Sometimes ties are unavoidable because two claimants have identical claims which SOMETIMES forces there to be a tie, and, thus, a need for a tie breaking system.

However, the situation is actually more subtle than that.

Sometimes ties occur with some of these methods by "accident."

Apportion:

h = 11; Total "population" = 1000

Claims:	A	B	C
	480	320	200

Take a few minutes to solve this using Hamilton's method; Webster's Method, and Jefferson's method.

Apportion:

h = 10; Total "population" = 1000

Claims:	A	B	C
	480	320	200

Take a few minutes to solve this using Hamilton's method; Webster's Method, and Jefferson's method.

Did anything unexpected happen here?

Without more theory than I have discussed it may not be clear what is going on!

So let me talk about the contribution of E.V. Huntington, and H. P. Young (Peyton Young) and Michel Balinski.

Huntington pioneered another approach to COMPUTING the solution to apportionment problems that involve Adams's, Dean's, Huntingon-Hill, Webster's, and D'Hondt's (Jefferson's) methods!!

Often it is helpful to have different algorithms for the same situation because for certain instances of the problem one algorithm may work more easily than another algorithm.

For example, finding the greatest common divisor of two numbers can be done with prime factorization or the Euclidean algorithm.

Find gcd of 3813 and 4371!

This approach I will call the table method but it is often described in the mathematical literature as a *rank-index* method.

Intuitive idea: Construct a table from the the claims data, a different table is needed for each of the 5 important divisor methods.

General procedure:

Largest number in table gets first seat; second largest second seat, etc.

If two states have the same number I will say the "share" the seats but strictly speaking what this means is that a randomization device is used to decide how to break the tie.

(Or one could use some other method to break ties - for example, order of population size.)

Example: Suppose h = 9 and 4 seats have been assigned but the next largest number in the table occurs 3 times. I will say seats 5, 6, and 7 are shared, and now one continues to try to assign seats in order of size of entries in the table. (Note: no tie breaking system needed.)

Example: Suppose h = 6 and 4 seats have been assigned but the next largest number in the table occurs 3 times. I will say seats 5, 6 are shared) and which of the three tied states will get seats 5 and 6 depends on the tie breaking system for the three tied states.

How does one prepare the table for Jefferson/D'Hondt? Starting with the initial claims construct a table the lines of which are the original data divided by: 1, 2, 3, 4, ...

Entries in tables will usually be shown with rounded to one decimal place. This rounding off may make the row sums be off. Some entries should involve repeated decimals and some are irrational!

Apportion:

h = 10; Total "population" = 1000

Claims:	A	B	C
	480	320	200

Claims: A B C

480 320 200

h = 9 and 10

	A		B		C	
Given data:	480		320		200	
Divide by: 1	480	1	320	2	200	4
Divide by: 2	240	3	160	5-6	100	9
Divide by: 3	160	5-6	106.7	8	66.7	
Divide by: 4	120	7	80		50	
Divide by: 5	96	10	64		40	

Need more lines if h =11!!!

Table with more lines:

	A	B	C
Given data:	480	320	200
Divide by 1	480	320	200
Divide by 2	240	160	100
Divide by 3	160	106.7	66.7
Divide by 4	120	80	50
Divide by 5	96	64	40
Divide by 6	80	53.3	33.3
Divide by 7	68.6	45.7	28.6

Why? The next largest number may not be in the current table but would be in the next line of a longer table!

	A	B	C
Given data:	480	320	200
Divide by 1	480	320	200
Divide by 2	240	160	100
Divide by 3	160	106.7	66.7
Divide by 4	120	80	50
Divide by 5	96	64	40
Divide by 6	80	53.3	33.3
Divide by 7	68.6	45.7	28.6

Webster: h = 12

	A		B		C	
Given data:		480		320		200
Divide by 1	480	1	320	2	200	3
Divide by 3	160	4	106.7	5	66.7	8
Divide by 5	96	6	64	9	40	13
Divide by 7	68.6	7	45.7	11	28.6	
Divide by 9	53.3	10	35.6		22.2	
Divide by 11	43.6	12	29.1		18.2	
Divide by 13	36.9		24.6		15.4	

For h = 13; C gets a 3rd seat!

How do we prepare the table for the other methods?

Webster:

First row, divide by 1/2
Second row, divide by 3/2
Third row, divide by 5/2

In general a + 1/2 for 0, 1, 2, 3, ...

Comment since the numbers in the table don't matter as long as the size order is the same for two tables, in practice one divides by: 1, 3, 5, 7, ... which avoids using the fractions 1/2, 3/2, 5/2, etc.

Adams:

First row, divide by 0
Second row, divide by 1
Third row, divide by 2
Fourth row, divide by 3

Of course we can't divide by zero so put an ∞ symbol for all entries in the first row!!!

In practical terms this means that if the house size is big enough all of the states AUTOMATICALLY get one seat as required by the Constitution. If there are 7 states, I will use the notation that the first 7 seats are shared!!! and if h is less than 7 we must use a tie breaking system.

Huntington-Hill, Dean's Method, and Adams automatically give one seat to each state if the house size is big enough!!

Jefferson and Webster have a US version and a European version, according as one forces each state to have one seat if the house size is large enough.

The European version of Jefferson is called D'Hondt; The European version of Webster is called St. Laguë.

Huntington-Hill:

First row, divide by 0
Second row, divide by $\sqrt{1x2}$
Third row, divide by $\sqrt{2x3}$
Fourth row, divide by $\sqrt{3x4}$

In general $\sqrt{a(a+1)}$ for 0, 1, 2, 3,....

Huntington-Hill is particularly important because it is the method that will be used by LAW to apportion the US House of Representative in after the 2020 Census is finished. Almost certainly NY will again lose seats after this census in Congress. This means required redistricting and perhaps gerrymandering!

Gerrymandering refers to the practice of redrawing districts to achieve some political or racial goal.

What might make one apportionment method better than another one?

Think of the different methods as optimizing some "quantity," and pick the method that does this?

Some notation:

Σ means sum; h = house size

Population state i: p_i

T = Total population = Σp_i

Seats assigned to state i: a_i (integer)

State i's fair share: $q_i = ((p_i)/T)h$

(sometimes called state i's quota)

Theorem: Hamilton's Method minimizes:

$$\Sigma\ |a_i - q_i|$$

and also:

$$\Sigma(a_i - q_i)^2$$

What one has done here is to set up a measure of comparison between different apportionments and found the method that with this measure is optimal.

This theorem looks at optimization at a global level.

With other objective functions for optimality, other apportionment emerge as optimal.

Huntington pioneered the use of a different approach to choosing among different apportionments.

His idea was to look at a pair of states and see if society was "happier" with the result when one transferred one seat from one state to the other!

When looking at whether a seat transfer improves things or makes things worse, one needs a way to measure how good things are in the current situation.

One can compare seat transfers between states from either a relative or absolute change point of view.

Reminder:

The relative change between two positive quantities a and b is given by:

$$(|a-b|)/\min(a, b)$$

while absolute change is given by:

$$|a-b|$$

The absolute value of a number is always positive.

$|-1-6| = 7$

$|-1+6| = 5$

$|6-1| = 5$

$|-6-1| = 7$

One could, but we will not, look at relative increase and/or decrease of two positive quantities.

Two natural ways to think about what a state received in an apportionment are:

representatives per person:

$$a_i/p_i$$

people per representative:

$$p_i/a_i$$

Now we can measure "disparity" between pairs of states in different ways:

Absolute difference between states j and i:

$$|a_j/p_j - a_i/p_i|$$

or

$$|p_j/a_j - p_i/a_i|$$

Relative difference between states j and i:

$$(|a_j/p_j - a_i/p_i|)/\min(a_j/p_j, a_i/p_i)$$

or

$$(|p_j/a_j - p_i/a_i|)/\min(p_j/a_j, p_i/a_i)$$

Perhaps unintuitively, there are usually DIFFERENT apportionments that MUST be chosen if you want to be fair with these different "measures" of fairness, even though a_j/p_j and p_j/a_j are reciprocals of each other!

Webster is the method which is best measured by:

$$|a_j/p_j - a_i/p_i|$$

but when one relativizes this:

$$(|a_j/p_j - a_i/p_i|)/\min(a_j/p_j, a_i/p_i)$$

Huntington-Hill is the best choice!

What are the pros and cons of these different methods? Isn't it somewhat arbitrary which of the 5 rank index (rounding rule equivalents) methods that picks?

Yes and no! But feelings run high.

Fact: If one uses a RELATIVE measure of fairness in comparing two states with regard to who get the next seat, then Huntington proved that Huntington-Hill is mandatory in all cases.

However, in addition to the issue of whether one should measure fairness in absolute or relative terms, there is another important idea to discuss.

Referred to as BIAS.

For a single instance of an apportionment problem many times the 5 methods agree, but sometimes they differ. However, perhaps when the same method is used over and over again for MANY instances there is a systematic way that one method discriminates against states in some way, in particular, due to their population size!

Perhaps one method is BIASED for against SMALL POPULATION states or LARGE POPULATION states.

Looked at in terms of rounding for numbers which lie between consecutive integers a and (a+1) we have this ordering:

Generous in giving an extra seat for a fraction:

Adams
Dean
Huntington-Hill
Webster
Jefferson

There is wide agreement that Adams consistently rewards smaller states with more seats than they "deserve," while Jefferson (D'Hondt" rewards large states with more seats than they "deserve."

Things get heated between what happens for Webster and Huntington-Hall.

The two leading experts on apportionment Balinski and Young argue that Webster is less biased than Huntington-Hill. Others are not sure. In the US apportionment problem the constitution is biased towards small states, in the sense that one can't not give each state one seat no matter how small its population!

I drove 200 miles from NYC at 36 mph and made the return trip at 48 miles mph? What was my average speed for the whole trip?

No, the answer is not 42 mph!

It is:

41.142857.....

Harmonic mean of a and b:

$$\frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Harmonic mean of a and (a+1):

$$\frac{2}{\frac{1}{a}+\frac{1}{a+1}}$$

$$\frac{2}{\frac{1}{a}+\frac{1}{a+1}}=\frac{2a(a+1)}{2a+1}$$

In general the harmonic mean of a and a+1 is smaller than the geometric mean of a and a+1, which in turn is small than the arithmetic mean of a and a+1!

The US Supreme Court has decided a variety of apportionment cases.

In the most important they decided not to agree when Montana went from having 2 seats in the House of Representatives to only 1 seat, that Dean's Method should be used because with Dean's Method Montana would have gotten 2 seats!

Other cases have dealt with the fact that Congress has given over the "mechanics" of carrying out the apportionment to the Commerce Department (which carries out the US Census) and that they use Huntington-Hill.

In the European context of apportionment problems, countries that use D'Hondt method tend to have more "stability" than those that use some of the other methods. Why? The party with the largest vote may get an "assist" by rewarding it with "more seats," which makes it easier to form stable coalitions in parliament. Not all scholars agree!

Other apportionment methods use a different rounding rule:

Jefferson (D'Hondt): Always round down

Adams: Always round up

Dean: Round based on the harmonic mean

Huntington/Hill (Currently the method used in America): Round using the geometry mean

Geometric mean of a and b equals square root (ab)

Geometric mean of a and b:

$$\sqrt{ab}$$

Geometric mean of a and (a+1):

$$\sqrt{a(a+1)}$$

I drove 200 miles from NYC at 36 mph and made the return trip at 48 miles mph? What was my average speed for the whole trip?

No, the answer is not 42 mph!

It is:

41.142857.....

Harmonic mean of a and b:

$$\frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Harmonic mean of a and (a+1):

$$\frac{2}{\frac{1}{a}+\frac{1}{a+1}}$$

$$\frac{2}{\frac{1}{a}+\frac{1}{a+1}}=\frac{2a(a+1)}{2a+1}$$

How does one decide which of these methods is better or worse?

What fairness properties do they obey?

Fairness axioms:

1. House size h monotonicity.

For many practical problems not important and there are algorithms that obey house size monotonicity and other critical fairness properties.

2. Population monotonicity

3. Quota

A state should get its fair share if it is an integer and the fair share rounded up or down to the next integer if it is not an integer.

1,593,436/35 = 45526.743 assigns too many seats.

Table uses: 46842 to assign 35 seats.

Party (State)	Vote (population)	Exact quota (share) of house	Webster or Huntington-Hlll number of seats
A	70,653	1.552	2
B	117,404	2.579	3
C	210,923	4.633	5
D	1,194,456	26.236	25
Total	1,593,436	35	35

Balinski-Young Theorem:

Michel Balinski (1933-2019)
H.P. (Peyton) Young (now at Oxford)

(who both taught for many years at CUNY's Graduate Center)

There is no apportionment method which obeys both population monotonicity and quota!!

(Core message: no "perfect" way to solve the apportionment problem.)

Have a good week!

Questions: email me at:

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and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html