The Talmudic Algorithm (2021)

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The "Talmudic Method" is the name sometimes given to the ideas that Michael Maschler (now deceased) and Robert Aumann used to explain the solutions to some problems that appear in the Babylonian Talmud. The solutions in the Talmud puzzled both religious scholars and mathematicians for many years.

The method, when applied to two claimants, is sometimes called the *contested garment rule* and more recently concede and divide. The set-up is this: There are two claimants A and B who have verified claims against an endowment or estate E whose size is less than that of A and B combined. Here is a specific example which gives the ideas that can be developed for the general case.

Example 1 (Contested garment rule):

A has a claim of 70; B has a claim of 90.

E = 120

Solution approach 1:

Note that the total claims, 70 + 90 = 160 exceed E =120, so we have a bankruptcy problem.

A says to the "judge" who distributed the endowment funds: B is only claiming 90 of the 120 that are being distributed, so 120 (the size of E) - 90 = 30 must be mine.

So A has an "uncontested" claim of 120 - 90 = 30.

B says to the "judge" who distributed the funds: A is only claiming 70 of the 120 that are being distributed, so 120 (the size of E) - 70 = 50 must be mine.

So A has an "uncontested" claim of 120 - 70 = 50.

We can give each claimant his/her uncontested claim. But what to do with the "undistributed" funds?

These uncontested claims amount to 50 + 30 = 80. Since E is 120, and 120 - 80 = 40, the judge divides this 40 of the remaining amount ("jointly contested claims") equally. This gives 40/2 =20 to each A and B.

Thus A gets 30 (uncontested) + 20 (equally divided contested amount) = 50.

B gets 50 (uncontested) + 20 (equally divided contested amounts) = 70.

Note that as a check, A + B = 120 = E as required.

Solution approach 2.

We will divide the original claims in half. We will use the constrained equality of gain method to try to meet these claims. If this can be done so there is money in E left over, we try to meet the remaining half of the claims by using the constrained equality of loss method. The idea between these two constrained equality (of gain or loss) is to try to treat the claimants as equally as possible, but for gains not to exceed their claims, and for losses not to "subsidize" the estate by adding money of their own to make it possible to achieve equality of loss.

So half the claim of A of 70 is 35 and half the claim of B of 90 is 45.

Now we have claims of 35 and 45 to be met using E = 120, using constrained equality of gain.

Thus, we can fully meet these claims: A gets 35 and B gets 45. (Had E been 75, A would get 35 and B would get 40.)

At this stage, since E is 120, and we have given A and B collectively 35 + 45 = 80, we have 120 - 80 = 40 to try to equalize the losses of A and B with regard to the remaining half of their claims.

A has a claim of 35 and B has a claim of 45, to be met using E* (remaining funds) = 40.

Since B at this stage has lost 45 and A only 35, to equalize losses we must first assign 10 units to B. This reduces B to the same level of loss as A, namely, 35. Now we want to treat A and B equally, so we give A and B equal amounts of 15 ((40-10)/2).

So in all A gets 15 and B gets 10 + 15 = 25.

Combined with A's constrained equality assignment of 35 that gives 50, and combined with B's constrained equality assignment of 45 gives 70. Thus, A gets 50 and B gets 70, which are the same amounts as we saw earlier!

The reasoning of Solution Method 1 cannot be extended to more than 2 claimants but the reasoning of Solution Method 2 can be extended to more than 2 players, as was shown by Michael Maschler and Robert Aumann (who won the Nobel Memorial Prize in Economics for his work in game theory).

One of the lovely features of the Talmudic Method is that when there are more than two claimants, the amount of money given to a subset of the claimants based on their original claims, using the amount of money that is assigned by the Talmudic Method is the same as when the Talmudic Method is applied to this smaller group of claimants with the endowment reduced by what these claimants got initially. (Thus, the amount assigned is "independent" of what the other claimants walked away with.) When a bankruptcy method obeys this rule, it is said to be *consistent*. I will now show an example of the Talmudic Method applied to three claimants and verify that it deals "consistently" with the claims for this example.

Example 2:

A claims 150; B claims 180; C claims 210. The endowment E = 360.

Solution:

Step 1: Try to meet half the claims using the constrained equality of gain method.

Half the claims:

A's half claim = 75; B's half claim is 90; C's half claim is 105. E = 360.

Since 75 + 90 + 105 is 270, we can meet these half claims with 360 - 270 = 90 units to spare.

Step 2: Try to meet the remaining half of the claims using constrained equality of loss.

Half the claims are:

A = 75; B = 90; C = 105; The remaining endowment is 90.

To equalize loss, currently A's loss is 75, B's loss is 90, and C's loss is 105; we must reduce the loss of C to 90 by giving C 15 units. We still have endowment left so we continue to give B and C equal amounts till we reduce their losses until they are both at the level of A's loss (75). This is accomplished by giving 15 units to B and 15 units to C (beyond the 15 he so far has gotten). So with 30 additional units we now have A, B, and C at a loss level of 75. But we still have endowment left: 90 - 15 - 2(15) = 45. We can now equally reduce the loss of each of the three claimants by giving 15 units to each of the claimants.

Thus, we have from Step 2:

A gets 15; B gets 15 + 15 = 30; C gets 15 + 15 + 15 = 45.

Combining the assignments with those from Step 1, we have:

A gets 75 + 15 = 90; B gets 90 + 30 = 120; C gets 105 + 45 = 150.

As a check we see that 90 + 120 + 150 = 360.

Now, let us verify that "consistency" holds. This requires 3 calculations.

Calculation C: C leaves

What do A (claim 150) and B (claim 180) get when C walks away with 150 from the 360?

Calculation B: B leaves

What do A (claim 150) and C (claim 210) get when B walks away with 120 from the 360?

Calculation A: A leaves

What do B (claim 180) and C (claim 210)) get when A walks away with 90 from the 360?

Here are the calculations in the same order as above.

Calculation: C

What do A (claim 150) and B (claim 180) get when C walks away with 150 from the 360?

If C leaves with 150 units, then the new endowment is E* = 360 - 150 = 210.

So we have the following problem to solve by applying the Talmudic Method (Contested Garment Rule in the case of 2 claimants):

A claims 150; B claims 180; E = 210.

Here are the details for this case; you can check the details for the others:

A's uncontested claim against B is 30.

B's uncontested claim against A is 60.

The uncontested amount remaining is 120.

So A gets 30 + (120)/2 = 90.

B gets 60 + (120)/2 = 120.

Thus, A gets 90 and B gets 120 of the E = 210.

However, this is exactly what A and B got when C was available as a claimant for the 360 units with a claim of 210!

Calculation B:

What do A (claim 150) and C (claim 210) get when B walks away with 120 from the 360?

A's claim 150; C's claim 210; E = 240

A gets 30 + 120/2 = 90.

C gets 90 + 120/2 = 150.

which agrees with what A and C got when B was present with a claim of 120 and E was 360.

Calculation A: A leaves

What do B (claim 180) and C (claim 210)) get when A walks away with 90 from the 360?

B's claim 180; C's claim 210; E = 270

B gets 60 + 120/2 = 120.

C gets 90 + 120/2 = 150.

which agrees with what B and C got when A was present with a claim of 150 and E was 360.

The Talmudic Method gives students a wide variety of opportunities to do calculations which require logical thinking in many interesting contextual situations which all lead to the bankruptcy model: bankruptcy, tax collection, estate allocation, and many others. It also allows one to represent many of the methods of solving bankruptcy questions using ideas from graphing of lines - analytical geometry. The mathematics of bankruptcy is a very appealing topic.

Bibliography:

Aumann, R., and M. Maschler, Game Theoretic Analysis of a Bankruptcy Problem from the Talmud, J. of Economic Theory, 36 (1985) 195-213.

O'Neill, B., A Problem of Rights Arbitration from the Talmud, Mathematical Social Sciences 2 (1982) 345-371.

Thomson, W., Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey, Mathematical Social Sciences 45 (2003) 249-297.

Young, H., Equity: In Theory and Practice, Princeton U. Press, Princeton, 1995.