

Motion Diagram (2022)

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The idea behind a motion diagram is to try to get insight into the "stable" or equilibrium values in a zero-sum or non-zero-sum matrix game by using a digraph (directed graph) to analyze what might be a good way to play the game.

Consider this matrix game (not zero-sum) which offers Row two actions (ways to play) and Column three actions (ways to play):

	Column I	Column II	Column III
Row 1	(2,3)	(6, -2)	(4, -3)
Row 2	(1,4)	(-4, 7)	(5,8)

The Row player picks one of two rows and the Column player picks one of three columns, and their independent choices determine the payoff to each. For example, the play (Row 2, Col II) yields Row a loss of 4 and Column a gain of 7.

Suppose that Row and Column get in the habit of playing in this pattern ((Row 2, Col II)) and Row asks him/herself can I do BETTER if Column continues to play Col II? The answer is "yes." Moving (motion) from Row 2 to Row 1 while Column continues to play Col II means Row will improve (and, but this is incidental to the analysis, Column will do worse). This observation will be

reflected in the drawing of a directed edge in the graph below (Figure 1) between the vertex representing (Row 2, Col II) and (Row 1, Col II). The arrow points away from the poorer outcome and towards the better outcome.

Now let us ask, if from the outcome (Row 1, Col II) if Column has any incentive to move from playing Col II. Column has two actions that might yield an improvement: motion to cell (Row 1, Col I) or the motion to (Row 1, Col III). Moving to play Col I would increase Column's payoff from -2 to 3, while moving to play Col III would decrease Column's payoff from -2 to -3. Thus, in the motion diagram there is an arrow leaving cell (6, -2) and directed into the dot (vertex) which represents cell (2, 3). (While there is NO directed edge in the motion diagram from the (6, -2) towards the (4, -3) dot, there is an edge going out of (4, -3) towards (6, -2) because if Row sticks to Row 1, Column can get a higher payoff by moving (motion) from the outcome (4, -3) to the outcome (6, -2). Note that there is ALSO an arrow going out of the dot representing (4, -3) to the dot representing the cell (2, 3) even though the columns I and III are not next to each other.

The arrows in Figure 1 are filled in using an analysis similar to the sample of arrows described above. Note that two vertices in Figure 1 show arrows only coming in: the dot representing (Row 1, Col I) and the dot representing the vertex corresponding to (Row 2, Col III). Thus, this game has two stable points corresponding to two pure strategy Nash equilibria. (There might be additional "mixed strategy" Nash equilibria.)

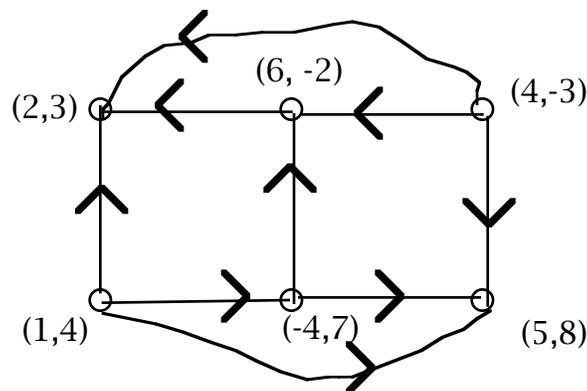


Figure 1 (Motion diagram corresponding to the 2x3 matrix game described. The numbers express the payoffs to Row and Column in that order. Thus (-4,7) is a payoff of -4 to Row and a payoff of 7 to Column when Row plays Row 2 and Column plays Col II.)

We can use the language of invalence (indegree) and outvalence (outdegree)

to help us see what is going on. Vertex (2,3) has invalence 3 and outvalence 0; vertex (6,-2) has invalence 2 and outvalence 1; vertex (4,-3) has invalence 0 and outvalence 3; vertex (1,4) has outvalence 3 and invalence 0; vertex (-4, 7) has invalence 1 and outvalence 2; vertex (5,8) has invalence 3 and outvalence 0. Those vertices with outvalence 0 show no incentive for either player to "move" and, thus, represent stable points and are pure strategy Nash equilibria for this game. Note there might also be mixed strategy Nash equilibria as well.

Sometimes vertices with outvalence 0 are called sinks and sometimes vertices of invalence 0 are called sources. This terminology is often used in situations where digraphs are applied to get insight into the flow of messages or fluids through a pipe network or a communications network which is modeled using a directed graph. Note that each directed edge is counted twice, in the invalence and outvalence counts, once at each end of the directed edge. The sum of the invalences equals the sum of the outvalences equals the number of directed edges in the digraph.

Motion diagrams are a relatively simple way to get insight into the dynamics of the way that a matrix game might be played when many rounds of play occur. If there are no stable points, then it follows by the existence of Nash equilibria for such games that the only equilibria must be mixed strategy equilibria. This is because any game of this kind must have pure strategy equilibria, mixed strategy equilibria or both.