Notes Inspired by Common Errors on Homework Problems: Game Theory (2022)

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Game theory questions

1. When one is solving a 2x2 zero-sum or non-zero-sum game, one typically needs a randomization device for optimal play. We learned how to design such a device only for 2x2 zero-sum games. (Here, 2x2 means 2 player with two actions available to each player.) The schematic diagram in Figure 1 shows a spinner that Column could use if he/she wanted to play column I, 1/4 of the time, and column II, 3/4's of the time. When one spins the arrow, it lands in the region labeled I, 1/4 of the time and the region labeled II, 3/4's of the time. If the spinner lands on the "lines," spin it again. For a given game which admits a mixed strategy equilibrium, one needs to design, based on the payoff matrix, a spinner for Row and a spinner for Column - two spinners.

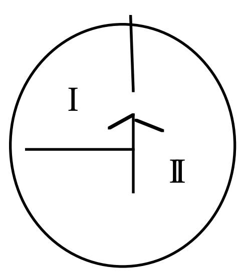

Figure 1

2. Zero-sum games such as the 2x2 game shown (Figure 2) are usually displayed with payoffs from the Row player's point of view. If Row plays Row 1 and Column plays column II then Row loses 5 units and Column gains 5 units.

	Column I	Column II
Row 1	8	-5
Row 2	-2	1

Figure 2

This game (Figure 2) has no dominating row and no dominating column. One has to check for both dominating rows and dominating columns independently. In a 6x7 game there might not initially be any dominating row but there might be a dominating column and after "crossing out" the DOMINATED column, there might NOW be a dominating row. So looking for dominating rows and columns in zero-sum games is a good first approach to "simplifying'" what is going on. One crosses out the rows or columns that give LESS favorable outcomes regardless of what one's opponent does. (If some row (or column) has some LESS favorable outcomes and the payoffs of other outcomes are tied regardless of what one's opponent does, one can also cross these rows (or columns) out.) Motion diagrams can be used for either zero-sum or non-zero sum matrix games.

This game (Figure 2) has no PURE STRATEGY Nash equilibrium so it must have a mixed strategy equilibrium. A very general theorem (John Nash): Any game of pure information must have Nash equilibria that are either pure strategy equilibria, mixed strategy equilibria, or the game can have both kinds of equilibria.

What we can find out about the game in Figure 2 is:

a. How Row should design a randomization device (spinner) to play optimally.

b. How Column should design an optimization device (spinner) to play optimally.

c. The expected value (if the game is played many times, the mean (average) payoff per play of the game to each of the players). There is an expected value for Row and an expected value for Column. This number is 0 for a fair game and if the expected value for Row is X (not zero), the expected value for column is -X. This is what zero-sum means: payoffs to the players add to zero. One can also think about what it would mean for a non-zero-sum 2x2 game that is played many time to be fair.

We have been studying 2x2 games in the context that players don't communicate with each other as to how they plan to play the game. The games are non-cooperative. For the non-zero-sum case, when the game is repeated many times, one can learn about one's opponent from the way they person plays. Some such games have the property that the expected value for both players is the same. When games only have positive or non-negative payoffs they have a different flavor from those where "optimal" play means losses for both players. A game may have a pure strategy Nash equilibrium where both players have negative payoffs. As we have seen, however, there might be Pareto improvement in such a game.

To design the Row spinner for the game in Figure 2, solve the equation (*). This equation uses the theorem that when Row plays optimally, it does matter how Column plays, so the payoff to Row when Column always plays column I equals the payoff to Row when Column always plays column II.

$8p - 2(1-p) = -5p + (1-p)$ (*)

where p represents the percentage of time to play Row 1 (p must be a number satisfying $0 < p < 1$).

Note: a **common error** here is to use the row entries as coefficients instead of the column entries.

To design the spinner for Column (again Figure 2) solve the equation (**). This equation uses the theorem that when Column plays optimally, it doesn't matter how Row plays, so the payoff to Column when Row always plays Row 1 equals the payoff to Column when Row always plays Row 2.

$8q - 5(1 - q) = -2q + (1 - q)$ (**)

where q represents the percentage of the time to play column I

q is a probability and must be a number satisfying $0<q<1$ (you should not get 0, 1 or a negative answer when the 2x2 game has not PURE strategy equilibrium).

NOTE: To find the EXPECTED value to Row, say, substitute the value of p into either the left or right side of (*). (If you do both calculations you should get the same number!) Using the fact that the game is zero-sum you can also get the answer by substituting q into the left or right side of (**)

Remember that when there is no pure strategy for a 2x2 zero-sum game, the expected value of the game takes the form:

E (expected value) = K(p-a)(q-b) + V (***)

Here K is a constant, p the percentage of the time Row plays Row 1, q the percentage of the time Column plays column I, a and b are positive and between 0 and 1, and V is the "value" of the game from Row's point of view. When V is not zero, Column's "value" is the negative of this number. When V is positive the game is to Row's advantage and when V is negative the game is to Column's advantage.

(Spoiler alert)

p = 3/16; q = 1/8; Expected value for Row is -1/8. This game is UNFAIR to Row. On average Row LOSES 1/8 every time the game is played. Since this is a zero-sum game, on average Column WINS 1/8 every time the game is played. Note each time the game is played the amounts which can change hands are 8, 5, 2, or 1. The number 1/8 is an "average" which only emerges over MANY plays to the game. Row should play Row 2, 15/16 of the time; Column should play Column II 5/8 of the time. If the game is played 32 times with EITHER player using their optimal strategy Row will lose -4 units on average and Column will gain +4 units on average. Remember the meaning of "equilibrium" here is that if either player deviates from their optimal play, the expected value does not change. This follows from (***).

3. Another zero-sum game:

	Column I	Column II
Row 1	9	-4
Row 2	7	1

Figure 3

There are no row dominations. However, column II always yields a better outcome for Column whichever row, the Row player chooses. SO we cross out Column I. (We cross out the dominated column, not the dominating column.)

If Row knows column I will never be played, Row's best response is always to play row 2. A this stage we cross out row 1 since it is dominated. We are left with a single cell with a 1. So this game is UNFAIR. When played well, on every play Row will win 1 and Column will lose 1. Should Column deviate from playing column 2, Row will win 7 and Column will lose 7, a much worse outcome.

4. We have learned how to play 2x2 zero-sum games, and using dominating strategy analysis we can sometimes simplify 2-person mxn zero-sum matrix games to a 2x2 or smaller case. Furthermore, if we are VERY lucky there may be one or more stable points (found using a motion diagram) for such zero-sum games. When there are several table points, they must have the same value. Stable points are "pure strategy" Nash equilibria. Remember Nash's Theorem applies to zero-sum as well as non-zero-sum matrix games.

5. The game below is a non-zero-sum game. In a pair such as (-12, 8) the first number is the payoff to Row and the second number is the payoff to Column.

	Column I	Column II
Row 1	(7,7)	(-12, 8)
Row 2	(8, -12)	(-1, -1)

Figure 4

Dominating strategy analysis applies to non-zero sum games, too. In this game Row 2 dominates row 1 and column II dominates column I but the result is a negative payoff for both players. Playing Row 1 and Column I gives both players a positive outcome, and represents a Pareto improvement over the (-1, -1) outcome. However, when the game is played, a fixed finite number of times between the same two players in an "experimental" situation, it is not unusual for the players to "lock" into the (-1,-1) outcome. This paradoxical game is sometimes called "Prisoner's dilemma." It has one PURE STRATEGY Nash Equilibrium, playing row 2 and column II. Check that it has no mixed strategy Nash equilibrium.

6. If one interprets the next game matrix (non-zero sum) either as ordinal outcomes, 4 being the best outcome, 3 being the next best outcome,...., 1 being the worst outcome, or as cardinal numbers (4 is bigger than 2, 3 or 1), one sees that the game has two PURE STRATEGY equilibria. Row 1 and Column II and Row 2 and Column I.

	Column I	Column II
Row 1	(3,3)	(2, 4)
Row 2	(4, 2)	(1, 1)

Figure 5

This is an example of the game of Chicken. One should also check for the possibility that this game (the cardinal version) has a mixed strategy equilibrium. Note that outcome (3,3) may seem appealing but is not stable, in the sense that both players have the incentive to "deviate" from actions that lead to this outcome.

Spoiler alert:

The mixed strategy equilibrium involves Row playing row 1 and row 2 each with probability 1/2 and Column playing column I and column II each with probability 1/2. The payoff to Row will be 5/2 and the payoff to Column will be 5/2.

7. As a check on using Baldwin's method for a ranked ballot election (sequential run-off based on the Borda Count), if there is a Condorcet winner, Baldwin's method will elect that candidate. Coombs is a run-off system based on the LARGEST number of LAST place votes. Standard run-off eliminates all but the top two first place vote getters. When there are ties, break them in a way that you explicitly state. One way, is to use a randomization procedure.

8. Remember that coins can be "biased" in many ways. Using a coin with heads coming up 3/4 of the time and tails coming up 1/4 of the time is a way to be "random." But for 2x2 zero-sum games with no pure strategy Nash equilibrium, there is only one optimal spinner for each of Row and Column.

10. To compute the median of a collection of numbers, sort them in increasing order and if there is an odd number of numbers, the median is the number in the "middle." Median "approval" points for a ballot with an add number of choices, where A, B, and E are the choices median or above, and the ballot got 7 votes, results in A getting 7 points, B getting 7 points and E getting 7 points. This is repeated for all the ballots and candidate with the highest sum for all of the ballots wins. One can also use this approach to RANK the candidates rather than to get a single winner.

11. Elections counting methods, Borda, Baldwin, plurality, etc. can be used to pick a single winner or to RANK all of the candidates (perhaps with ties).