Notes for Remote Presentation 13:

Game Theory/Fairness Modeling

April 25, 2021

Gale/Shapley Matching

New Topic:

Market design ideas which basically go back to the ideas of the great mathematicians/game theorists:

David Gale

Lloyd Shapley

Two-sided Markets

(Model originally due to David Gale and Lloyd Shapley (Nobel memorial prize in Economics) - now both deceased)

Sometimes described under the title stable matchings and Gale/Shapley.

Other major contributors:

Alvin Roth (Nobel memorial prize)
Robert Irving
David Manlove

The paper that started it all!

College admissions and the stability of marriage

D. Gale; L. S. Shapley. The American Mathematical Monthly, Vol. 69, No. 1 (Jan., 1962), 9-15.

Monthly not associated with FIRST PUBLICATION of profound mathematics.

Major applications:

1. College admissions

2. Matching hospitals with residencies to medical school graduates

3. Matching court justices and clerks

4. School choice

5. Kidney exchange

Vanilla (Basic) Rules:

Ladies rank the men without ties
Men rank the ladies without ties

All individuals would rather be paired than be "single."

Rankings are ordinal

Extensions exist where:

a. players on either side of the market might prefer to be unmatched rather than having a specific player paired with them.

b. players can rank the players on the other side of the market with ties allowed

There are different ways of representing the information given by the players in a pairing problem which searches for stable pairings.

Columns can be ranks - entries are players.

Columns can be the people - entries are the ranks.

Problem instance: 4x4 example

4 men rank 4 women without ties

	1 st	2nd	3rd	4th
m1	w1	w2	w3	w4
m2	w2	w1	w4	w3
m3	w1	w4	w2	w3
m4	w2	w3	w4	w1

4 women rank 4 men without ties

	1st	2nd	3rd	4th
w1	m4	m3	m2	m1
w2	m4	m3	m1	m2
w3	m2	m1	m4	m3
w4	m4	m2	m3	m1

Table instance

Men's preferences: (example: w3 is m2's 2nd choice)

	w1	w2	w3	w4
m1	1	4	3	2
m2	1	3	2	4
m3	2	1	4	3
m4	2	3	4	1

Women's preferences: (example m1 is w4's 4th (last) choice)

	m1	m2	m3	m4
w1	4	1	2	3
w2	3	1	4	2
w3	4	3	2	1
w4	4	2	3	1

One sided markets:

Situations where two sets are paired but one of the sets has no "feelings" about the assignment!

Example: Students attending a mathematics research program at a college are provided dorm rooms. The students may have preferences about the rooms but the rooms don't have preferences about the students.

Examples: Students who participated in a mathematics contest are each awarded a different book as a prize for having gotten honorable mention in the contest.

The students may have preferences about the books they get but the books don't have preferences about who owns them.

One approach to such problems that some might view as fair is to assign the rooms or prizes to the individuals at random. However, it might happen that each player gets his/her worst choice. Random may be fair but it is rarely is "efficient" in making the recipients happy.

Goal: Stable matching

What does this mean?

Suppose (m,w) are paired in some matching. w can't find another man m* who she prefers to m and where m* prefers w to the woman m* is paired with in the matching!

Similarly for m in this pair.

Such a pair is called a blocking pair.

When a matching M has no blocking pair it is called stable.

When a matching M has a blocking pair (m, w) one or both members of this pair have incentive to work outside the system and pair up with their "better" choice.

Gale and Shapley showed there is a simple algorithm to guarantee that there is at least one stable marriage. Usually there are MANY stable pairs.

Unintuitively, among all the stable marriages there is one which is BEST from the ladies point of view and one which is BEST from the men's point of view - though sometimes these two coincide in which case there is ONLY one stable marriage for the preferences involved.

These stable marriages are known as the MALE OPTIMAL stable marriage and the FEMALE OPTIMAL stable marriage.

Assume the FEMALE and MALE optimal stable marriages are different.

The FEMALE optimal stable marriage is the worst from the male point of view among all stable marriages.

The MALE optimal stable marriage is the worst from the female point of view among all stable marriages.

If n women and n men rate each other there are n! possible matchings, for a given problem instance. The number of stable marriages can be as few as 1 for an instance but there is a family of marriage problem instances where the number of stable marriages grows exponentially in n. Exactly how large this number can be is still an *unsolved* problem.

Men's preferences:

	w1	w2	w3	w4
m1	1	4	3	2
m2	1	3	2	4
m3	2	1	4	3
m4	2	3	4	1

Women's preferences:

	m1	m2	m3	m4
w1	4	1	2	3
w2	3	1	4	2
w3	4	3	2	1
w4	4	2	3	1

The Gale-Shapley deferred acceptance algorithm proceeds in rounds.

The FEMALE optimal stable marriage is produced when the girls PROPOSE to the boys; The MALE optimal marriage occurs when the boys propose to the girls.

Female optimal:

Set up: Each boy is at a table in front of the "gym."

In each round any un-paired girl proposes to the next boy on her preference list who has not already turned her down!

To propose go to the table of the next person on your list.

When one or more girls propose to a boy he temporarily picks the one who is ranked highest on this list. If in a future round a better choice arrives he (perhaps) temporarily changes to this better choice.

(For the male optimal matching reverse the roles of girls and boys - boys propose.)

Again:

When girls propose they do best.

When boys propose they do best.

Best, in the sense of getting their highest ranked person of the opposite sex they can in ANY stable pairing.

Men's preferences: (example: w2 is m3's first choice)

	w1	w2	w3	w4
m1	1	4	3	2
m2	1	3	2	4
m3	2	1	4	3
m4	2	3	4	1

Women's preferences: (m1 is w4's 4th choice

	m1	m2	m3	m4
w1	4	1	2	3
w2	3	1	4	2
w3	4	3	2	1
w4	4	2	3	1

Round 1: m1 m2 m3 m4
(all) w1,w2 w3,w4

m2 accepts w1; m4 accepts w4; w2 and w3 moves to Round 2

Men's preferences:

	w1	w2	w3	w4
m1	1	4	3	2
m2	1	3	2	4
m3	2	1	4	3
m4	2	3	4	1

Women's preferences:

	m1	m2	m3	m4
w1	4	1	2	3
w2	3	1	4	2
w3	4	3	2	1
w4	4	2	3	1

Round 2: m1 (w2,w3); m2 w1 m3 w3 m4 w2,w4

m2 accepts w1; m3 accepts w3; m4 accepts w4; w2 to Round 3

Men's preferences:

	w1	w2	w3	w4
m1	1	4	3	2
m2	1	3	2	4
m3	2	1	4	3
m4	2	3	4	1

Women's preferences:

	m1	m2	m3	m4
w1	4	1	2	3
w2	3	1	4	2
w3	4	3	2	1
w4	4	2	3	1

Round 3: m1 m2 m3 m4

(w2); w2 w1 w3 w4

m1 accepts w2; m2 accepts w1; m3 accepts 3, m4 accepts w4

Men's preferences:

	w1	w2	w3	w4
m1	1	4	3	2
m2	1	3	2	4
m3	2	1	4	3
m4	2	3	4	1

Women's preferences:

	m1	m2	m3	m4
w1	4	1	2	3
w2	3	1	4	2
w3	4	3	2	1
w4	4	2	3	1

Stable:	m1	m2	m3	m4
	w2	w1	w3	w4

To find the female or male optimal marriage is not HARD but requires careful coordination between the information generated as the algorithm is carried out and the use of the two preferences tables to carry out the algorithm! When done by hand it requires careful practice.

In the medical residency version the algorithm that is used it is students propose but in the past the hospital's optimal version of the algorithm was used!

In Great Britain, the same problem arises and they did not use a stable matching algorithm and the market *unraveled* - that is, people worked outside the centralized system to find pairings.

This suggests that using an algorithm that finds stable solutions is valuable and wise.

Much research is done to try to find stable matchings efficiently that lie "between" the female and male optimal stable marriages. The idea is to find a "median" stable matching. This problem is often computationally hard and also hard to define the extent to which the collective happiness improves.

Also note that in problems involving markets with many participants, it is impractical for participants to rank all of the players on the other side of the market.

The NYC School Choice matching procedure is Gale/Shapley in "spirit" but limits the number of schools that one can list (rank).

Still many open questions with respect to both theoretical and applied aspects of stable marriage ideas.

In particular might some stable matching which lies "between" female optimal and male optimal matching be "wise" when there are many stable marriages?

One-side market

Room allocation problem

Assigning prizes to contest winners

To solve the room assignment problem one can use an algorithm called the *top trading cycle* algorithm.

Assume the rooms have been assigned at random (or some other method) and then for convenience number the players by the room number they are given to occupy.

Now get the preferences of each player for the rooms.

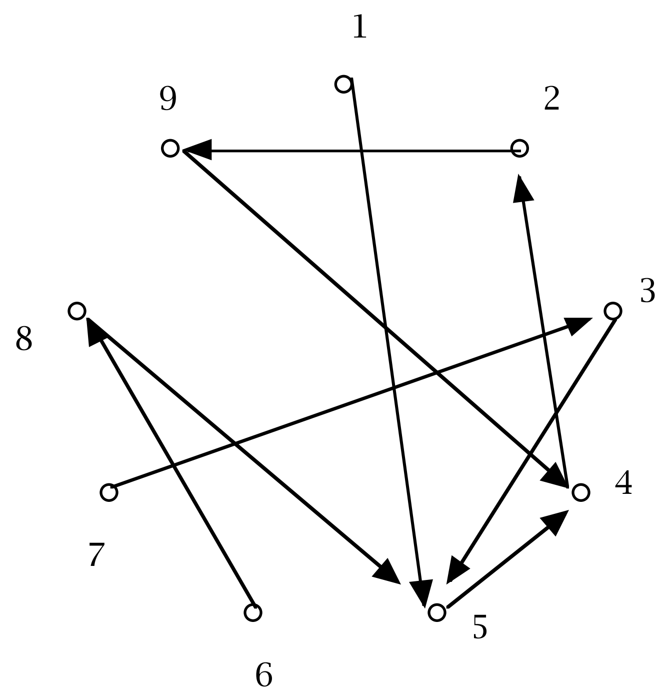
	1st	2nd	3rd	4th	5th	6th	7th	8th	9th
1	5	4	7	8	9	3	2	6	1
2	9	6	3	8	1	4	7	5	2
3	5	8	2	3	6	4	9	1	7
4	2	6	8	3	7	9	1	5	4
5	4	9	6	5	1	3	7	2	8
6	8	9	7	5	1	4	2	3	6
7	3	1	6	9	7	5	8	2	4
8	5	1	3	2	9	8	4	6	7
9	4	6	8	1	3	7	2	9	5

Players are numbered so that they occupy room with their number - that is, Player 4 is currently in room 4. Player 4's favorite room is 2.

	1st	2nd	3rd	4th	5th	6th	7th	8th	9th
1	5	4	7	8	9	3	2	6	1
2	9	6	3	8	1	4	7	5	2
3	5	8	2	3	6	4	9	1	7
4	2	6	8	3	7	9	1	5	4
5	4	9	6	5	1	3	7	2	8
6	8	9	7	5	1	4	2	3	6
7	3	1	6	9	7	5	8	2	4
8	5	1	3	2	9	8	4	6	7
9	4	6	8	1	3	7	2	9	5

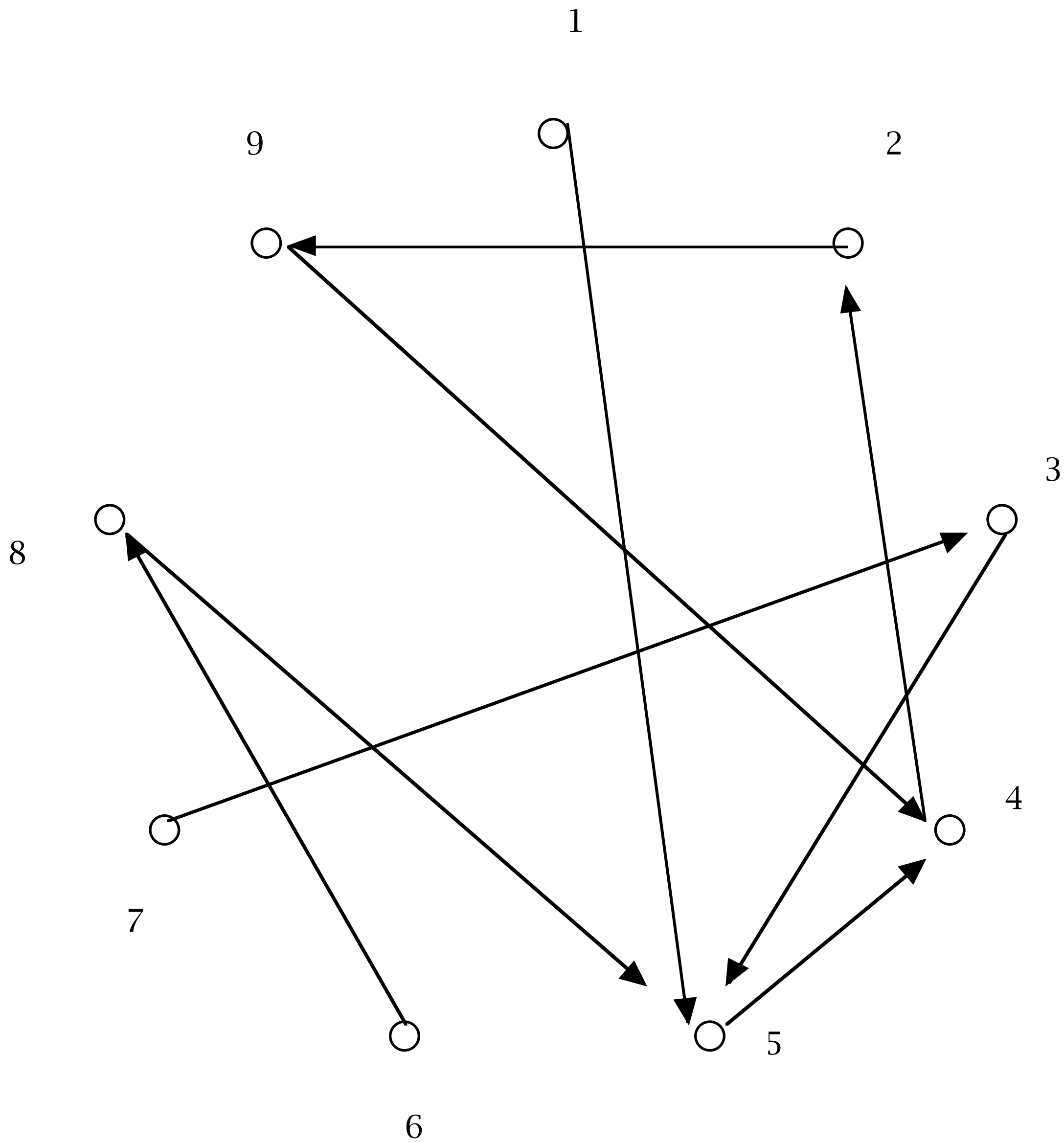
Construct a directed graph by drawing a directed edge from each player to his/her most preferred room. There will be a "self-loop" a vertex joined to itself if a player prefers his/her current room best.

Theorem: The graph constructed in this way always has at least one directed circuit.



Assign Player 2 to room 9; Player 9 to room 4, and Player 4 to room 2.

Delete these players and repeat.


1
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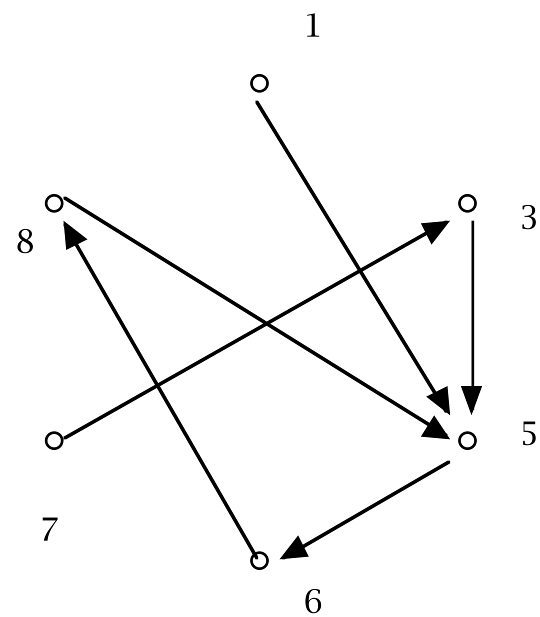
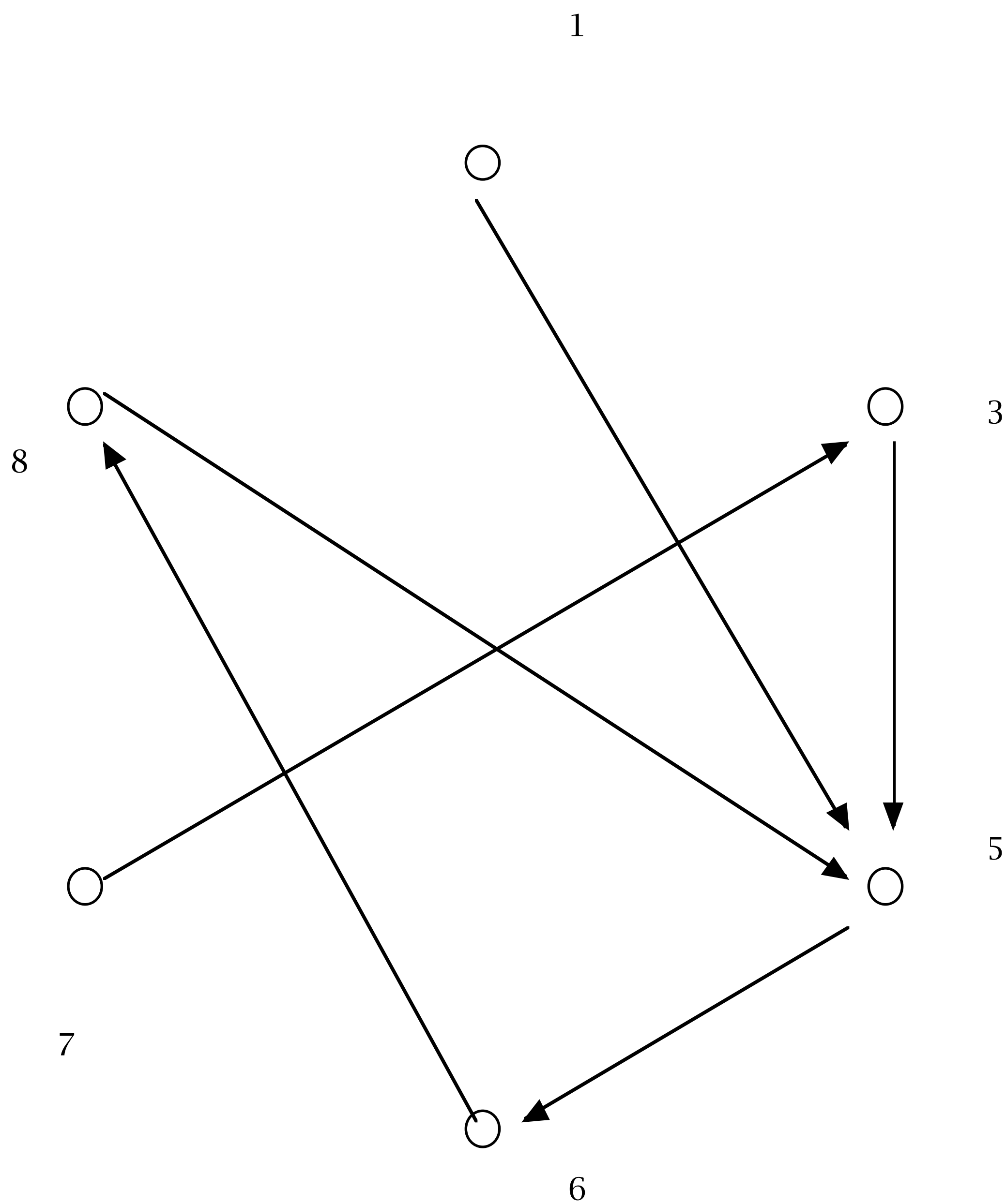
Remaining 6 players point to their favorite room which is still available for assignment.



Player 5 gets room 6; Player 6 gets room 8; player 8 gets room 5. Remove these three players. Repeat.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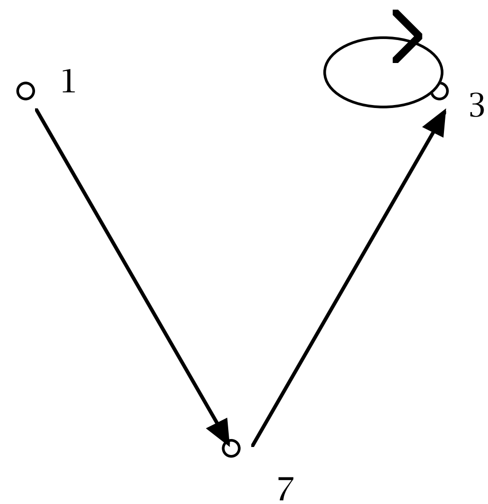
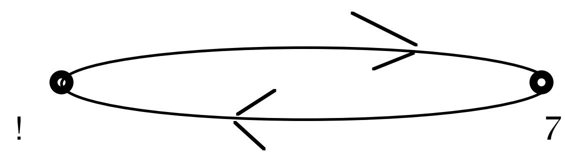


Player 3 gets Room 3;

In the last Round:



Player 1 gets room 7; Player 7 gets room 1.

Algorithm terminates!

Ideas related to this approach (Top Trading cycles) and the deferred acceptance algorithm have been used in school choice algorithms, in a variety of cities, including, the use of ideas of Alvin Roth and his doctoral students for the NYC school choice program. Boston at one time used this top trading cycle approach to school choice.

Mathematics education:

Mathematics education issues:

a. What ways are there to prevent the widespread mathophobia we see in America?

b. Are high stakes tests in the interests of stakeholders for mathematics education?

Stakeholders

Students; parents; teachers; school administrators; employers; state and local governments; text book publishers; mathematics community; computer science community, etc.

Curriculum

Current curriculum is heavily biased in favor of what has been done historically. Many new elementary tools for insights into the world such as basic graph theory are not currently taught? Why?

Balance between algorithms and skills and conceptual understanding

Theoretical vs. applied issues

Mathematical modeling

Historically, MAA generated reports through its CUPM Committee (Committee on the Undergraduate Program in Mathematics) which was issued every 10 years (print) and made recommendations for courses for mathematics majors and the mathematical preparation of teachers of mathematics.

Fact:

Surprisingly large amounts of new elementary mathematics is being discovered.

More emphasis should be given to this fact. We teach mathematics to convey knowledge from the past.

However, we also teach mathematics to generate more new mathematics and applications of mathematics for the future.

Have a good week!

Questions: email me at:

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and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html