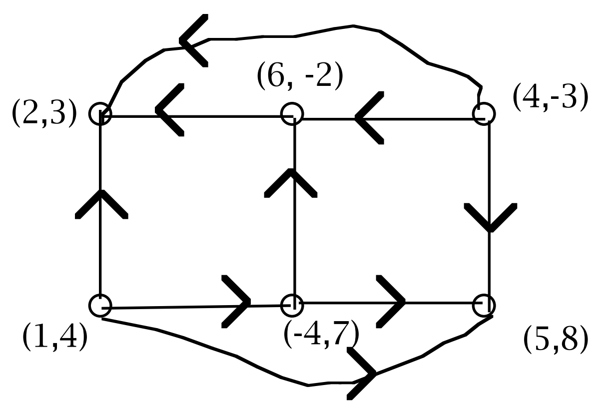
# Notes for Remote Presentation 4:

# Game Theory/Fairness Modeling

February 14, 2022

More complex motion diagram: (2x3 matrix game: 2 actions for Row; 3 actions for Column. Payoffs are ordered pairs - (Row's payoff,  Column's payoff). This game is not a zero-sum game.



# Find the optimal way to play by Row and Column:

| | Column I | Column II |
|---|---|---|
| Row 1 | 3 | -4 |
| Row 2 | -8 | 9 |

# How should one play this game? Is the game fair?

| | Column I | Column II |
|---|---|---|
| Row 1 | 2 | -4 |
| Row 2 | -6 | 12 |

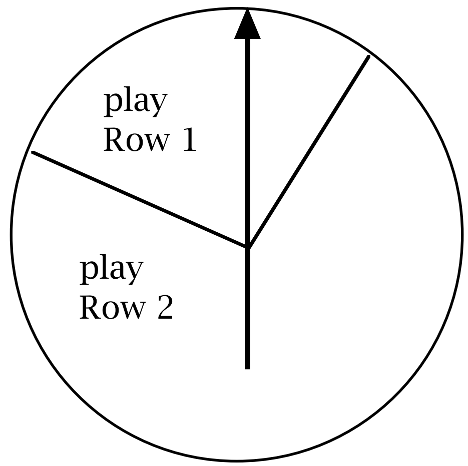
When Row plays Row 2 and Column plays Col I, Column wins 6 and Row loses 6.

There are no dominating rows or columns. The motion diagram is a cycle, and, hence, there is not "stable point" for this game.

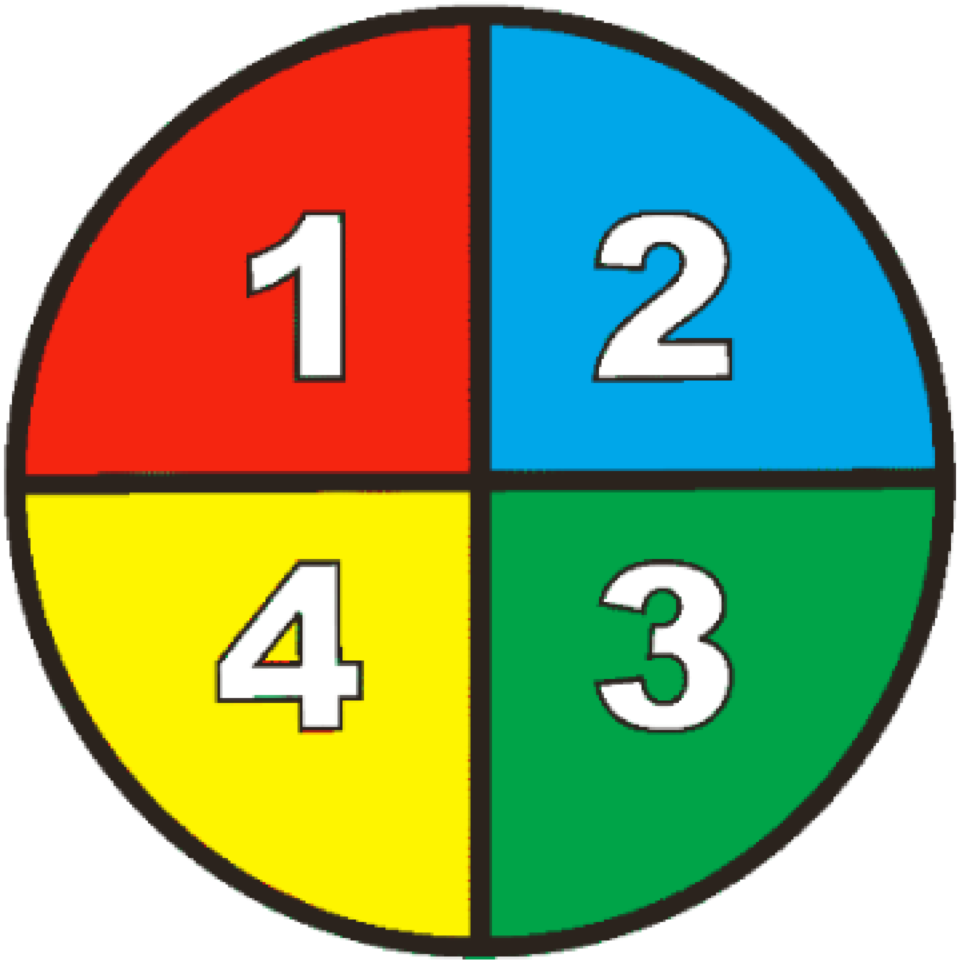
To play optimally Row/Column need to independently design a spinner, so that their random play is optimal - gives as good an expected value as possible for each of them.

Perhaps surprisingly, if this game is played many times over and over again, although no outcome from any particular play is 0, on the average neither player wins nor loses in the long run. The expected value (average payoff for each player over many plays of the game) is 0. The game is FAIR!!

# Row's spinner:



Spinner card 4 equal area regions:



We showed that for 2x2 when there is no domination and the motion diagram is a circuit, when Row played optimally it did not matter whether column always played Col I or always played Col II. The expected payoff would be the same!

| | Column I | Column II |
|---|---|---|
| Row 1<br><br>**Play with probability p** | 2 | -4 |
| Row 2<br><br>**Play with probability 1-p** | -6 | 12 |

EV: Col, always plays Col I: $2p - 6(1-p)$

EV: Col, always plays Col II: $-4p + 12(1-p)$

$$2p - 6 + 6p = -4p + 12 - 12p$$
$$8p - 6 = -16p + 12$$
$$24p = 18$$
$$p = 3/4 \quad (\text{Value: } 2(3/4) - 6(1/4) = 0) \text{ Fair!}$$

| | Column I<br><br>**Play with probability q** | Column II<br><br>**Play with probability 1-q** |
|---|---|---|
| Row 1 | 2 | -4 |
| Row 2 | -6 | 12 |

EV: Row, always plays Row 1: $2q - 4(1-q)$

EV: Col, always plays Row 2: $-6q + 12(1-q)$

$$2q - 4 + 4q = -6q + 12 - 12q$$

$$6q + 18q = 16$$

$q = 2/3$ (Value: $2(2/3) - 4(1/3) = 0$) Fair!

Note that the spinners for Row and Column are not the same in this example though they might be the same in some examples.

| | Column I | Column II |
|---|---|---|
| Row 1 | 2 | -4 |
| Row 2 | -6 | 12 |

Also, notice that: 2(12) - (-4)((-6) = 0

In this case, the fact is that the determinant of the 2x2 matrix is zero, and the game is fair. There is a general result here for 2x2 zero-sum games whose motion diagram is a circuit.

We can extend our look at two player games in perhaps an unexpected direction - decision making by a single player Row, where the role of Column is taken by a PASSIVE "player," to be called "nature."

Example 1: A farmer (Row) can take different actions, such as what crops to grow and the payoffs will depend on the weather that "nature" throws at the farmer, and, will affect the payoffs from different actions. However, nature does not try to make life hard for the farmer. The Column player is passive, and not actively opposing Row.

Example 2: Where should a power company drill for natural gas, with different payoffs depending on the state of "nature."

# Toy problem: (Payoffs in thousands of $'s)

| | I | II | III | IV | |
|---|---|---|---|---|---|
| 1 | 12 | -3 | -1 | 0 | |
| 2 | 3 | -1 | 11 | 2 | |
| 3 | 2 | -7 | 5 | 1 | |
| 4 | 30 | 2 | 1 | -11 | |
| | | | | | |

# What action would you recommend?

There is a single dominated row in the prior matrix. Here is the simplified decision matrix:

| | I | II | III | IV | |
|---|---|---|---|---|---|
| 1 | 12 | -3 | -1 | 0 | |
| 2 | 3 | -1 | 11 | 2 | |
| 3' = old 4 | 30 | 2 | 1 | -11 | |
| | | | | | |

What action would you pick?

Ideas:

Don't pick an action where you can get badly "hurt."

You tend to be a lucky person so choose an action with a high return?

Assume the states of natures have probabilities and maximize expected value.

| | I | II | III | IV | Worst in Row |
|---|---|---|---|---|---|
| 1 | 12 | -3 | -1 | 0 | -3 |
| 2 | 3 | -1 | 11 | 2 | -1 |
| 3' = old 4 | 30 | 2 | 1 | -11 | -11 |
| | | | | | |

Chose to play that row which makes the largest loss as small as possible. This is a maxi-min approach. -1 (Here: Row 2.)

Choose to play the row with the largest gain. (Here Row 3'.)

One could assume the Columns were equally likely to occur and try to maximize the expected value. This is equivalent to selecting the Row with the largest Row sum.

| | I p=1/4 | II p=1/4 | III p=1/4 | IV p=1/4 | Row Sum |
|---|---|---|---|---|---|
| 1 | 12 | -3 | -1 | 0 | 8 |
| 2 | 3 | -1 | 11 | 2 | 15 |
| 3' = old 4 | 30 | 2 | 1 | -11 | 22 |
| | | | | | |

In this case this would be Row 3.

Many theorists argue that using the "equi-probable" model is not wise. They argue that nearly always one has information already available (past records of the weather, say) or information that one could buy that would make it possible for one to assign BETTER probabilities for the states of nature. Then one uses these probabilities to compute the action with the best expected value, including the costs of getting better information.

Here probabilities (which sum to 1 as they should) have been assigned to the Columns, but they are not all equal to each other. What decision by Row is best, that is, which Row should row choose?

| | I p=1/12 | II p=1/6 | III p=0 | IV p=3/4 | Expected value |
|---|---|---|---|---|---|
| 1 | 12 | -3 | -1 | 0 | |
| 2 | 3 | -1 | 11 | 2 | |
| 3' = old 4 | 30 | 2 | 1 | -11 | |
| | | | | | |

| | I p=1/12 | II p=1/6 | III p=0 | IV p=3/4 | Expected value |
|---|---|---|---|---|---|
| 1 | 12 | -3 | -1 | 0 | 6/12 |
| 2 | 3 | -1 | 11 | 2 | 19/12 |
| 3' = old 4 | 30 | 2 | 1 | -11 | 65/12 |
| | | | | | |

With these probabilities the best expected value occurs for Row 3. (But note 1/4 of the time Row loses 11 units.)

Comment: One can easily design "toy" examples where different "appealing" approaches call for different behavior on the part of the decision maker (Row). No one approach to decision making is "always" the best, seemingly.

But wait, what about this idea suggested by the American mathematician/statistician Leonard Jimmie Savage (1917–1971):

After the action a decision maker takes some state of Nature occurs and the decision maker can assess with hindsight what would have been a better decision.

This suggests the idea of regret.

Suppose the state of nature that occurred turned out to be Column i

Let the maximum row value in Column i be M.

The regret for Row j is:

payoff in Row j - M

| | I p=1/12 | II p=1/6 | III p=0 | IV p=3/4 | |
|---|---|---|---|---|---|
| 1 | 12 | -3 | -1 | 0 | |
| 2 | 3 | -1 | 11 | 2 | |
| 3' = old 4 | 30 | 2 | 1 | -11 | |
| | | | | | |

# Regret matrix:

| | I | II | III | IV | Largest regret |
|---|---|---|---|---|---|
| 1 | 18 | 5 | -12 | 2 | 18 |
| 2 | 27 | 3 | 0 | 0 | 27 |
| 3' = old 4 | 0 | 0 | 10 | 13 | 13 |
| | | | | | |

# Third Row minimizes the max regret!

One could also minimize the expected regret or do various other calculations to decide what row to pick based on the regret matrix.

Mathematical analysis may not be able to tell you what to *do* but it can help clarify what choices you have and their consequences.

Let us move beyond zero-sum games. Such games are relatively "rare" in the real world restricted to the artificial world of matrix games with two players.

What about 2x2 non-zero sum games?

Can we give good advice about how to play such games as we could for zero-sum games?

Answer: Usually No!!!

Optimal play here?

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, 3) | (-6, 4) |
| Row 2 | (4, -6) | (-2, -2) |

| | Column I | Column II |
| --- | --- | --- |
| Row 1 | (3, 3) | (-6, 4) |
| Row 2 | (4, -6) | (-2, -2) |

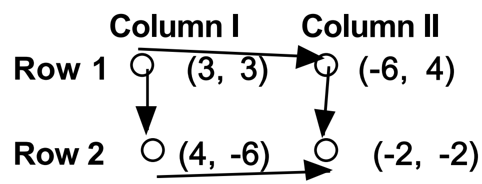
Note there is a dominating row (Row 2 dominates Row 1) and a dominating column (Column II dominates Column I) so "rational play" yields the result that both players lose 2 units every time the game is played! This does not seem like a "rational" outcome!!

| | Column I | Column II |
| --- | --- | --- |
| Row 1 | (3, 3) | (-6, 4) |
| Row 2 | (4, -6) | (-2, -2) |

The outcome (3,3) is better than (-2,-2). Playing Row 1 and Column I is not "*stable*," but BETTER! Playing Row 1 and Column I is known as a PARETO IMPROVEMENT over the (-2,-2) outcome. It is named for the economist Vilfredo Pareto (1848-1923 - born France, died in Switzerland) who called attention to this type of situation. If either player deviates from playing Row 1 and Column I while the other player does not, the result is that that the player who deviates does better and "punished" his/her opponent.

Even if the players agree to cooperate over multiple plays of this game there is incentive to break the agreement since if you "defect" from "cooperate" you do better and your opponent does worse!

Using a motion diagram one can see that there is one stable point (equilibrium) for this matrix:




# To achieve this stable outcome Row plays Row 2; Column plays Col II
# We say this is a pure strategy equilibrium - no randomization!

Here the only stable outcome is (-2, -2) and (3,3) is an UNSTABLE Pareto improvement.

Consider the "similar game"

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, 3) | (-3, 10) |
| Row 2 | (10, -3) | (-2, -2) |

If Row and Column can AGREE to alternately play: Row 2, Col I and Row 1, Col II then Row's cumulative payoff looks like:
10, 7, 17, 14, 24, 21, 31, .....

-3, 7, 4, 14, 11, 21, ....

# Brief review of what we have learned about games?

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, 1) | (-1, 4) |
| Row 2 | (-4, 0) | ( 2, -5) |

How would you play this game if you had to play it once?
How would you play this game if you had to play it many times?

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, 1) | (-1, 4) |
| Row 2 | (-4, 0) | ( 2, -5) |

Are there Row or Column dominations?

Are there pure strategy stable points? (Draw a motion diagram and check.)

***Theorem: (John Nash - Nash Equilibria)***

For a general class of games, for simplicity, say 2x2 matrix games, the game has:

a. A pure strategy equilibrium

or

b. A mixed strategy equilibibirum

or

c. Both both pure and mixed strategy equilibria

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, 1) | (-1, 4) |
| Row 2 | (-4, 0) | ( 2, -5) |

So there are no pure strategy Nash equilibria so we know (Nash's Theorem) that there most be a mixed strategy Nash equilibrium?

How do we find it?

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, 1) | (-1, 4) |
| Row 2 | (-4, 0) | ( 2, -5) |

## Row's game

| | Column I | Column II |
|---|---|---|
| Row 1 | 3 | -1 |
| Row 2 | -4 | 2 |

## Column's game

| | Column I | Column II |
|---|---|---|
| Row 1 | 1 | 4 |
| Row 2 | 0 | -5 |

Nash equilibrium:

Row wants an equalizing spinner for what Column can get in Column's game:

| | Column I | Column II |
|---|---|---|
| Row 1<br>p | 1 | 4 |
| Row 2<br>1-p | 0 | -5 |

$p = 4p-5(1-p)$

$p = 4p-5+5p$

$p = 5/8$ (with EV to Column of 5/8)

Columns wants an equalizing spinner for what Row can get in Row's game:

| | Column I<br>**q** | Column II<br>**1-q** |
|---|---|---|
| Row 1 | 3 | -1 |
| Row 2 | -4 | 2 |

$3q-1(1-q) = -4q+2(1-q)$

$3q-1+q=-4q+2-2q$

$10q=3$

$q=3/10$ (with EV to Row of 1/5.)

Students in your classes will want to know how 5/8 and 1/5 compare! Here, clearly 5/8 is larger but one has to subtract two fractions to see by how much!

| | Column I | Column II |
|---|---|---|
| Row 1 | (3, 1) | (-1, 4) |
| Row 2 | (-4, 0) | ( 2, -5) |

(3,1) is a better outcome for both players than (1/5,5/8) but (3,1) is unstable. (1/5,5/8) is the Nash equilibrium and stable.

Since in finding their Nash equilibrium strategies Row and Column don't directly use their own payoffs, other approaches to how to play a game like this "wisely," include playing ones Prudential strategy or Counter prudential strategy. So one approach the game above is to look for your payoffs from the 9 approaches (but of course there are many others one might consider) that involve Row and Column playing Nash, Prudential, and Counter prudential strategies:

Row's possible ways to try to do well:

Row can play:

1. Nash equilibrium strategy (there may be many)

2. Prudential strategy (optimal play in Row's game)

3. Counter prudential strategy (Best response to Column's Prudential strategy.)

Column's possible ways to try to do well:

Column can play:

1. Nash equilibrium strategy (there may be many)

2. Prudential strategy (optimal play in Column's game)

3. Counter prudential strategy (Best response to Row's Prudential strategy.)

## Row's game

| | Column I | Column II |
|---|---|---|
| Row 1 | 3 | -1 |
| Row 2 | -4 | 2 |

## Column's game

| | Column I | Column II |
|---|---|---|
| Row 1 | 1 | 4 |
| Row 2 | 0 | -5 |

Note: On the left payoffs are from Row's point of view and on the right from Column's point of view!!

Note: As a zero-sum game Row's game requires playing a mixed strategy. However, Column's game as a zero sum game has dominations.

Psychologists use game theory to try to understand human cooperation by doing experiments involving the play of these games.

Look at Free Lunch game essay on class web page.

Idea: Given the opportunity for a "cash infusion" if you ***share*** its value with your opponent, will you share it equally? If what your offer your opponent (say, 60% for you 40% for your opponent) is refused, you both get nothing!!!

Spoiler alert!!

Traditional game theory says if offered $100 to share you should keep $99.99 and offer your opponent .01! Rational behavior is that .01 makes you richer than refusing the offer by the "ungenerous" opponent.

Experiments show that what people actually do, varies with age, gender, religion, ethnicity!!

Give examples where voting and elections are used in ***YOUR*** life or those the lives of the students you teach.

What are the features (components) of an election or voting system that we are trying to understand elections so we can "improve" the way the election is carried out?

Components of an election or voting system:

1. One needs voters or decision makers.

Say n voters or decision makers. (I will assume n is odd but one needs ways to treat ties when they occur.)

2. One needs alternatives or candidates to vote on or choose from.

Say there are m candidates.

3. One needs a way for the voters to express their opinions about the choices or candidates.

The usual way this is done is by using a ***ballot***. We also need to think about how voters behave in filling out ballets.

4. Based on the ballots one needs a way of deciding who the winner or collection of winners is. Sometimes one is filling seats on a committee and there may be several people elected.

What types of ballots are you familiar with from elections in which you have participated?

In America we vote for:

* President

* Members of the House of Representative and Senate

* Governors

* Mayors

* Chairperson of a department

* Faculty committees

* Best actress

* Best movie

* Best rookie pitcher

* Best player in a particular football game

Mathematics has explored the surprisingly many ways to to construct ballots as inputs to elections.

The the major distinction parallels the two major kinds of numbers we use:

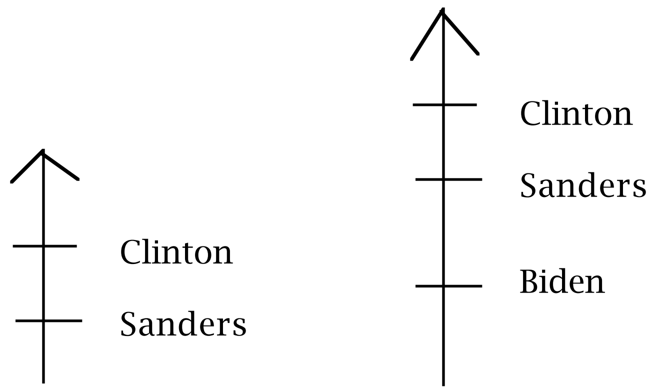
ordinal (counting numbers)

cardinal numbers (to measure)

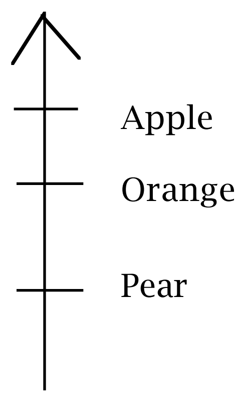
* ordinal or rank ballots (with or without ties)

Show order of the candidates (choices) but not how strongly one feels about the candidates.

# Two and three candidate ordinal ballots by individuals:

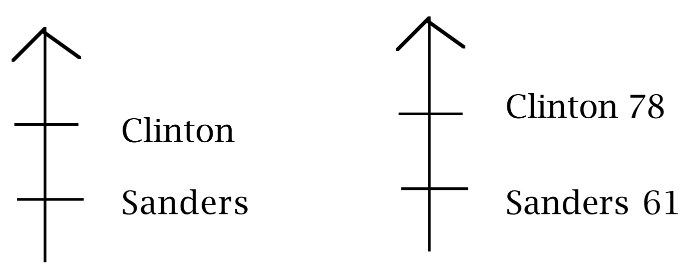



Here is a ranking of three fruits by a group of people:



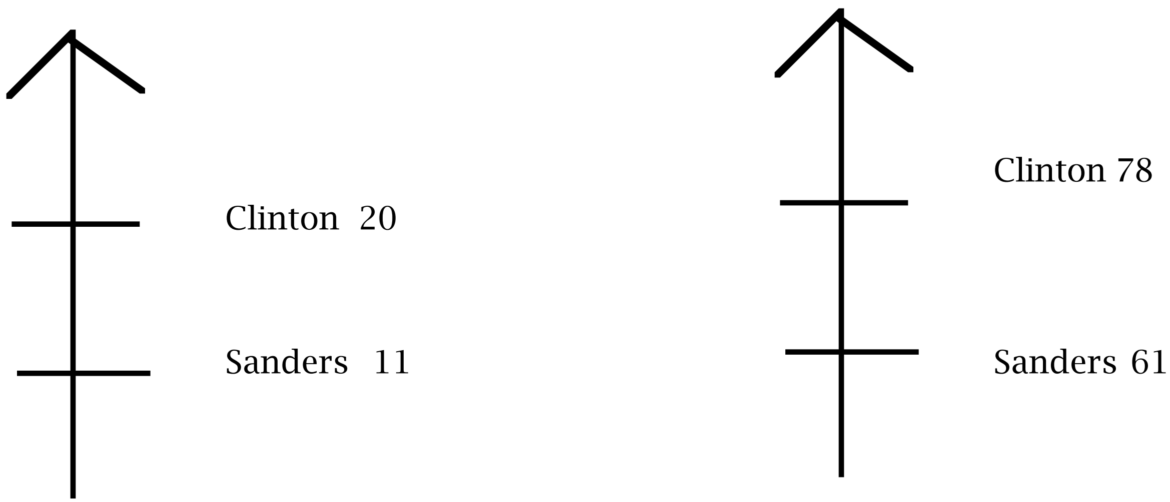
* cardinal ballots show intensity of support

Scale (100 high; zero low)



# Same ranking but very different information about "intensity."

Scale: 100 high; 0 low



Is ballot truncation allowed?

Truncation on a ballot refers to the voter not listing all of the candidates but only some of the candidates.

Truncation can occur because the voter chooses to only list candidates he/she knows information about. Sometimes the voter may not know anything about some available choice.

Sometimes a ballot is truncated because a voter knows what method is used to count the ballots, and voting for more than one candidate will help not only one's favorite but other choices as well. This is called ***strategic voting***. It involves lying about one's real preferences.

When voting strategically, one "lies" about one's true views about the candidates to help a particular candidate or group of candidates win. Thus, one might only rank "conservative" candidates in a primary election with many people seeking office.

In some elections truncation is not permitted! Sometimes voters must rank the candidates without ties.

*In Australia voting is required by law! (Being ill can be offered as an "excuse.")*

Other ballots:

* approval ballot

Only vote for those candidates you are willing to have serve

* yes/no voting

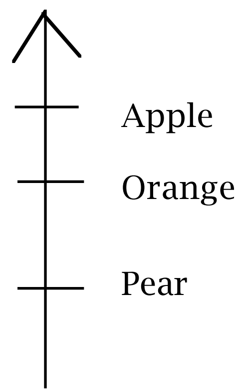
For each candidate you vote yes or no.

For many voting situations when voters don't want to rank or vote for against a particular candidate it is because they don't know anything about the candidate. They perhaps have never heard of this person.

Some ballots allow voters to divide candidates into two groups; those they know and don't. This separation can be done strategically.

This is very different from when they don't want to vote for a candidate(s) because they are nervous it will give help to candidates they prefer more to rank or vote for.

Mary's ballot for ranking the three fruits, apple, orange, pear:




A more word-processor friendly way to record Walter's preferences:

Apple>Pear>Orange

Here is an election involving three candidates using ordinal/ranked ballots. (Due to Kristopher Kunsterhjelm)

90 Voters

32 votes for: L>C>R
31 votes for R>C>L
26 votes for C>L>R
1 vote for C>L>R

Which choice should win the election? If you needed to rank the choices for the group what would the ranking be?

What method did you use and why?

Note conceptually:

We are seeking a FUNCTION (input output device) which inputs an "election" (a collection of ballots) and outputs:

a. Single winner (sometimes several winners)

or

b. A ranking for "society."

Voting methods:

1. Plurality voting

The winner is whoever gets the highest number of first place votes.

## 2. Run-off election

If no one gets a majority of the votes cast, eliminate all candidates except those two that got the largest number of first place votes and see who wins in this 2-way race.

3. Sequential run-off (sometimes called IRV, instantaneous run-off voting)

If no one has a majority eliminate the choice with the fewest first place votes. Transfer votes for the eliminated candidate to remaining candidates. Repeat until a single winner emerges.

4. Borda Count (can be used for ordinal ballots with ties allowed)

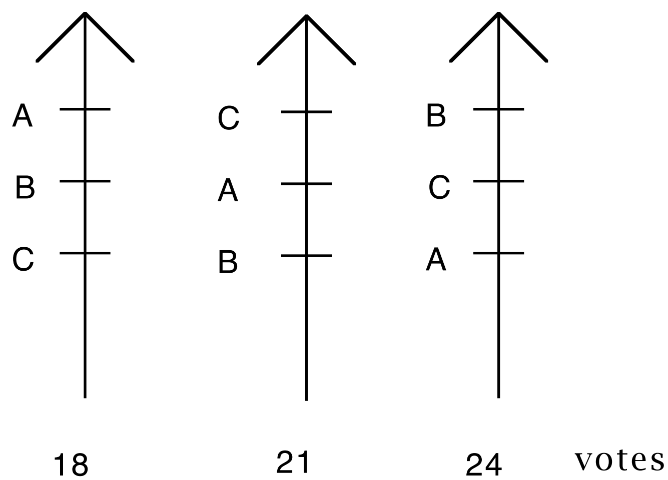
For each ballot, the contribution towards candidate i's "score" or "points" will be the number of candidates below i on the that ballot. Sum the scores (points) for the candidates for all the ballots, the winner being the candidate with the largest number of points (score).

## 5. Condorcet's method

That candidate who can beat all the others in a two-way race wins.

This is a very appealing method but unfortunately, despite the intuition that there should always be such a candidate, there are elections for which no Condorcet winner exists!

# No Condorcet winner:




A beats B 39 to 24
B beats C 42 to 21
C beats A 45 to 18

## 6. Baldwin's method

Conduct a sequential run-off election using the Borda Count.

The candidate with the smallest Borda Count is eliminated and that candidates votes are transferred to the other candidates. Borda count is recomputed and we repeated until there is a single winner.

Condorcet compliant methods are ones which elect a Condorcet winner when there is one but elects some winner for all the ballots that constitute the election.

Theorem: If there is a Condorcet winner Baldwin elects that candidate!

# 7. Bucklin's method

If no candidate has a majority of first place votes, add the number of first and second place votes for the candidates. Often now several candidates with have a majority but the winner will be the one with the largest majority. If no such candidate use 3rd place votes, etc.

# 8. Coomb's Method

If no candidate has a majority, eliminate the candidate with the largest number of *last place* votes and redistribute votes for that candidate, and repeat the process.

## 9. Baldwin's method

If no candidate has a majority of first place votes, eliminate those candidates whose Borda count is at or below the mean and redistribute votes for eliminated candidates. Repeat the procedure till a majority candidate emerges.

# 10. Median and above

For each ballot give one point to any candidate at the the median position or above. The candidate with the largest number of points wins.

# Have a good week!

Questions: email me at:

jmalkevitch@york.cuny.edu

and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html