Notes for in Class Presentation 6:

Game Theory/Fairness Modeling

February 28, 2022

Voting methods:

1. Plurality voting

The winner is whoever gets the highest number of first place votes.

2. Run-off election

If no one gets a majority of the votes cast, eliminate all candidates except those two that got the largest number of first place votes and see who wins in this 2-way race.

3. Sequential run-off (sometimes called IRV, instantaneous run-off voting)

If no one has a majority eliminate the choice with the fewest first place votes. Transfer votes for the eliminated candidate to remaining candidates. Repeat until a single winner emerges.

4. Borda Count (can be used for ordinal ballots with ties allowed)

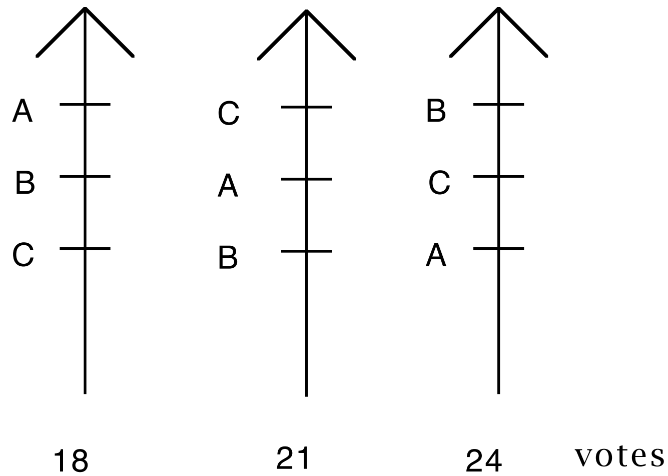
For each ballot, the contribution towards candidate i's "score" or "points" will be the number of candidates below i on the that ballot. Sum the scores (points) for the candidates for all the ballots, the winner being the candidate with the largest number of points (score).

5. Condorcet's method

That candidate who can beat all the others in a two-way race wins.

This is a very appealing method but unfortunately, despite the intuition that there should always be such a candidate, there are elections for which no Condorcet winner exists!

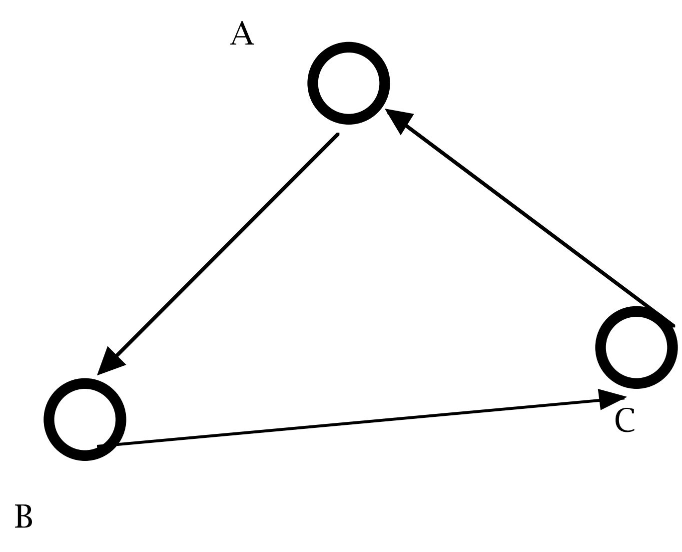
No Condorcet winner:



A beats B 39 to 24
B beats C 42 to 21
C beats A 45 to 18

A beats B in a two-way race;
B beats C in a two-way race;
C beats A in a two-way race.

No Condorcet winner!



6. Baldwin's method

Conduct a sequential run-off election using the Borda Count.

The candidate with the smallest Borda Count is eliminated and that candidates votes are transferred to the other candidates. Borda count is recomputed and we repeated until there is a single winner.

Condorcet compliant methods are ones which elect a Condorcet winner when there is one but elects some winner for all the ballots that constitute the election.

Theorem: If there is a Condorcet winner Baldwin elects that candidate!

7. Bucklin's method

If no candidate has a majority of first place votes, add the number of first and second place votes for the candidates. Often now several candidates with have a majority but the winner will be the one with the largest majority. If no such candidate use 3rd place votes, etc.

8. Coomb's Method

If no candidate has a majority, eliminate the candidate with the largest number of *last place* votes and redistribute votes for that candidate, and repeat the process.

9. Nanson's method

If no candidate has a majority of first place votes, eliminate those candidates whose Borda count is at or below the mean and redistribute votes for eliminated candidates. Repeat the procedure till a majority candidate emerges.

10. Median and above

For each ballot give one point to any candidate at the the median position or above. The candidate with the largest number of points wins.

11. Black's Method

If there is no Condorcet winner, then use the Borda Count

Axiomatics

Formal mathematical systems involve undefined terms and rules (axioms) that are assumed true, and then on deduces theorems.

For example, the standard version of Euclidean Geometry is due to David Hilbert

Hilbert's axioms (rules) consist of 20 assumptions by David Hilbert in 1899 in the book Grundlagen der Geometrie (The Foundations of Geometry) each of which is independent. (Independence means without this axiom there are other "geometries" other than Euclidean geometry that satisfy the axioms.

For example: without Triangle Congruence (which takes the form SAS side-angle-side in Hilbert) one gets what is called taxicab geometry. Taxicab distance is simpler than Euclidean distance because it only involves absolute value rather than square roots.

Taxicab distance between (a,b) and (c,d) is given by:

$$|a-c| + |b-d|$$

(Intuition: distance is measured along lines parallel to the right angle coordinate axes)

Note contrast with "crow flies" or Euclidean distance.

To choose between different methods of deciding elections we can look at fairness properties that the different methods obey. If method M obeys all the fairness rules we think are important we could try to change the way elections are run to this method.

Unfortunately, things are not that "simple."

Spoiler alert:

Arrow's Theorem:

For elections which involve ranked ballots (ties on the ballot allowed) with 3 or more candidates there is NO way to decide an election that obeys all the fairness conditions in a short list!

Kenneth Arrow, who attended City College (before it was part of CUNY) in NYC and got his doctorate at Columbia University won the Nobel Memorial Prize in Economics for this work!

Arrow's "fairness" conditions:

Voters produce ranked ballots with ties. One seeks a ranked ballot with ties for "society."

a. Procedure works for any ballots.
b. Non-dictatorial
c. Non-imposed
d. Monotone
e. (IIA) Independence of irrelevant choices

a. (Universal) No set of ballots is considered too "strange" to be considered for counting. The elections decision "committee" will not "edit/censure" a set of ballots.

c. (Imposed)

In ancient Greek some decisions were made based on the recommendation of a person "other than the decision" maker. Thus, one can consult the Delphi oracle, one's parents, a religious leader to make a decision. Arrow does not allow this!

d. (Monotone)

Having more voter support can't harm a candidate.

IRV (sequential run-off) and ordinary run-off are not monotone.

e. (IIA) Independence of irrelevant alternatives.

In deciding whether society prefers X to Y based on the ballots the answer should not depend of how the voters feel about alternative Z.

The Borda Count does not obey independence of irrelevant alternatives.

If in a restaurant you are offered the choice of ham or lamb as the ONLY and say you prefer ham. The waiter returns and says, by the way, chicken is also available. If you reply in that case I prefer to have the lamb rather than the ham, this might appear "strange."

Gibbard–Satterthwaite Theorem

Given an election involving 3 or more candidates the only decision method where lying about one's true opinions (strategic voting) about the candidates won't be advantageous for a voter is *dictatorship*!

Allan Gibbard (1942-) worked in the area of philosophy (but majored in mathematics at Swarthmore) and Mark Satterthwaite is an economist who teaches in a school of business.

Weighted voting:

One person, one vote is an important notion in election theory for the United States. (Except the electoral college does not work that way!) More educated or wealthy Americans still only get one vote for who should be President.

How should one have representation in an amalgamation of countries of different sizes such the European Union or the counties of upstate New York where there are towns of very different sizes in a typical county?

County governments in New York provide funds for roads, county police, preventing river/lake contamination, etc.

Sometimes to be fair, different players should have different influence so voting is done using the idea of weighted voting.

Some players cast a bigger vote than others. Thus, each player i will cast a block of votes.

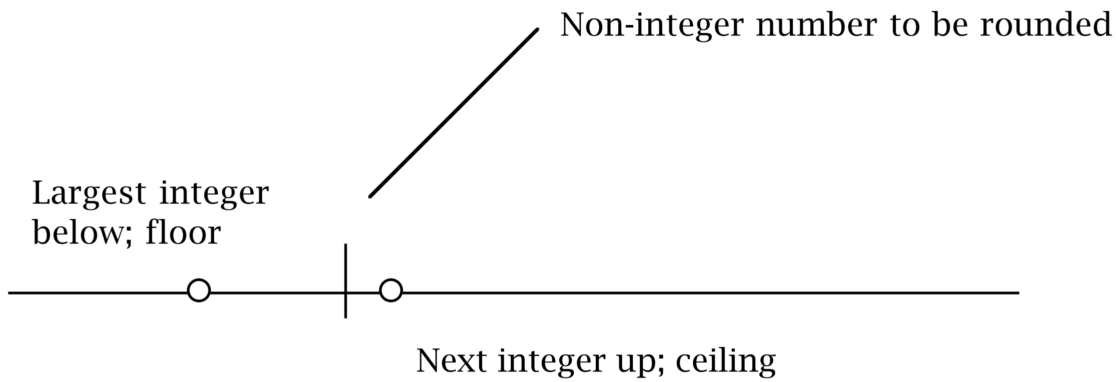
This leads to the idea of weighted voting. One has players 1, 2, 3,..., n each of whom casts a block of votes.

So player i casts w(i) votes, called i's weight.

A collection of players is called a *coalition*. To take action one needs to have sum of the weights of the players in a coalition exceed some number Q, called the Quota. Often the quota is set at the integer above 1/2(sum of weights) + 1

However, in some cases more than a simple majority is needed for action.

Rounding: Floor and ceiling functions: (Notation due to Donald Knuth, a computer scientist.)



$$\lceil 3.12 \rceil = 4 \text{ (ceiling)}$$

$$\lfloor 5.88 \rfloor = 5 \text{ (floor)}$$

Notation: players are "named" so that the small numbered players cast the largest votes (ties allowed).

Example:

[8; 6, 5, 2, 1]

Four players: player 1 casts 6 votes, player 2 casts 5 votes, player 3 casts 2 votes, and player 1 casts 1 vote.

[8; 6, 5, 2, 1]

For an action to be taken a group of "players" with at least weight 8 have to "work together." Thus, {1,2} is a coalition with players 1 and 2 and since 11 is bigger than 8, they can make sure that a "bill" passes the legislature. {1,2) is called a winning coalition.

[8; 6, 5, 2, 1]

{3,4} command only 3 points and so by themselves can't take action. This is a losing coalition.

A player is called a *veto* player if that player is a member of every *minimal winning* coalition.

That is, a group of players who if any player is deleted from that coalition can no longer take action. (Win)

Question: County Z has 5 towns with populations of 900,000, 500,000, 500,000, 400000, and 200,000. What might be a good set of weights and a quota for a county legislature with 5 players?

Weighted voting game:

$$[\ 13;\ 9,\ 5,\ 5,\ 4,\ 2]$$

Minimal winning coalitions:

{1, 2} {1, 3} {1, 4} {2, 3, 4}

Player 5 has NO power! Player 5 is never a member of any MINIMAL winning coalition. That is, a group of players who if any player is deleted from that coalition can no longer take action. (Win)

Example:

[5; 4, 3, 2] Three players named 1, 2, and 3 who cast 4, 3, and 2 votes respectively. The 5 is called the quota. Players with combined weight of 5 are needed to take action.

Is Player 1 twice as powerful as Player 3 because 4 is twice 2?

[5; 4, 3, 2]

Which coalitions (collections) of players can take action?

Minimal winning coalitions - no subset of a minimal winning (MW) coalition wins:

{1,2}, {1,3}, {2,3}

Given [5; 4, 3, 2], we have total symmetry here for the MW. The MW coalitions are:

{1,2}, {1,3}, {2,3}

so it should be apparent that in this game all three players have equal influence!!!

An isomorphic game would be:

[2; 1, 1, 1]

because its minimal winning coalitions are also:

$$\{1,2\}, \{1,3\}, \{2,3\}$$

Power indices: (Variants differ in using all winning versus MW coalitions) (Name are not standardized.)

a. Coleman

b. Banzhaf

c. Shapely

d. Deegan-Packel-Johnston

[5; 4, 3, 2]

MW: {1,2}, {1,3}, {2,3}

Coleman:

1 is in two coalitions
2 is in two coalitions
3 is in two coalitions

So 1 has 2/6 as a power; 2 has 2/6 as a power; 3 has 2/6 as a power!

Look at the pattern of Yes and No votes of the 3 players:

YYY wins
YYN wins
YNY wins
YNN loses
NYY wins
NYN loses
NNY loses
NNN loses

Underlines show when a Yes changed to a No changes a win to a loss. So of the underlined items each player has 2 out of a total of 6. (This is Banzhaf Power.)

So each player has equal Banzhaf power.

Note: We only look for "pivots/swing," that is changes when a sequence of Y's and N's wins, and changing a Y to an N makes a win a loss.

It turns out that looking at situations where a pattern yields a loss and changing a No to a Yes wins, just doubles the number of pivots/swings because we are computing a ratio.

Shapley-Shubik Power Index

[5; 4, 3, 2]

1 *2* 3
1 *3* 2
2 *1* 3
2 *3* 1
3 *1* 2
3 *2* 1

Pivot player is shown in italics - second in every case for this example.

Hence:

Player 1 has 2 pivots out of 6; power 1/3

Player 2 has 2 pivots out of 6; power 1/3

Player 3 has 2 pivots out of 6; power 1/3

Remember that 2/6 is the same fraction as 1/3.

Banzhaf was not trained as a mathematician. He was trained in the law. He is most famous for winning cases against tobacco companies that smoking is harmful to one's health.

He also won a Supreme Court decision which overturned the use of weighted voting in Nassau County because there were players with NO Power!

Nassau and Suffolk now have legislatures rather than weighted voting but most upstate NY Counties have weighted voting procedures.

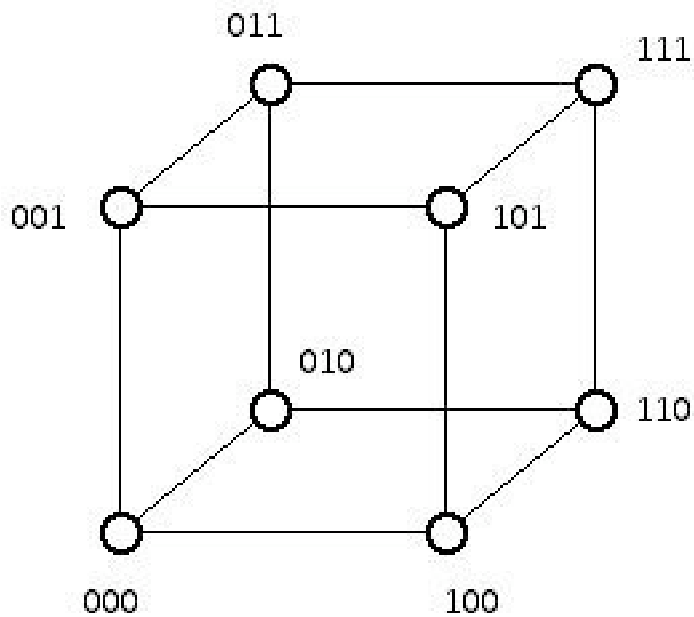
In NYS weights must be assigned to the players in the weighted voting games for county governments so that the Banzhaf Power is proportional to the population of the players involved.

Pattern of Yes/No for lines in a Banzhaf power table for 3-players "corresponds" to the labels needed for a 3-dimensional cube:

NNY,YNY,NYY,YYY(top)

NNN,YNN,NYN,YYN(bottom)

Think of N as a 0 and Y as a 1:



Three-cube made from two 2-cubes! Top layer all entries end in 1; bottom layer all entries end in 0!

Banzhaf table for 4 players correspond is obtained by pasting together two copies of a 3-cube to get a combinatorial 4-cube.

Have a good week!

Questions: email me at:

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and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html