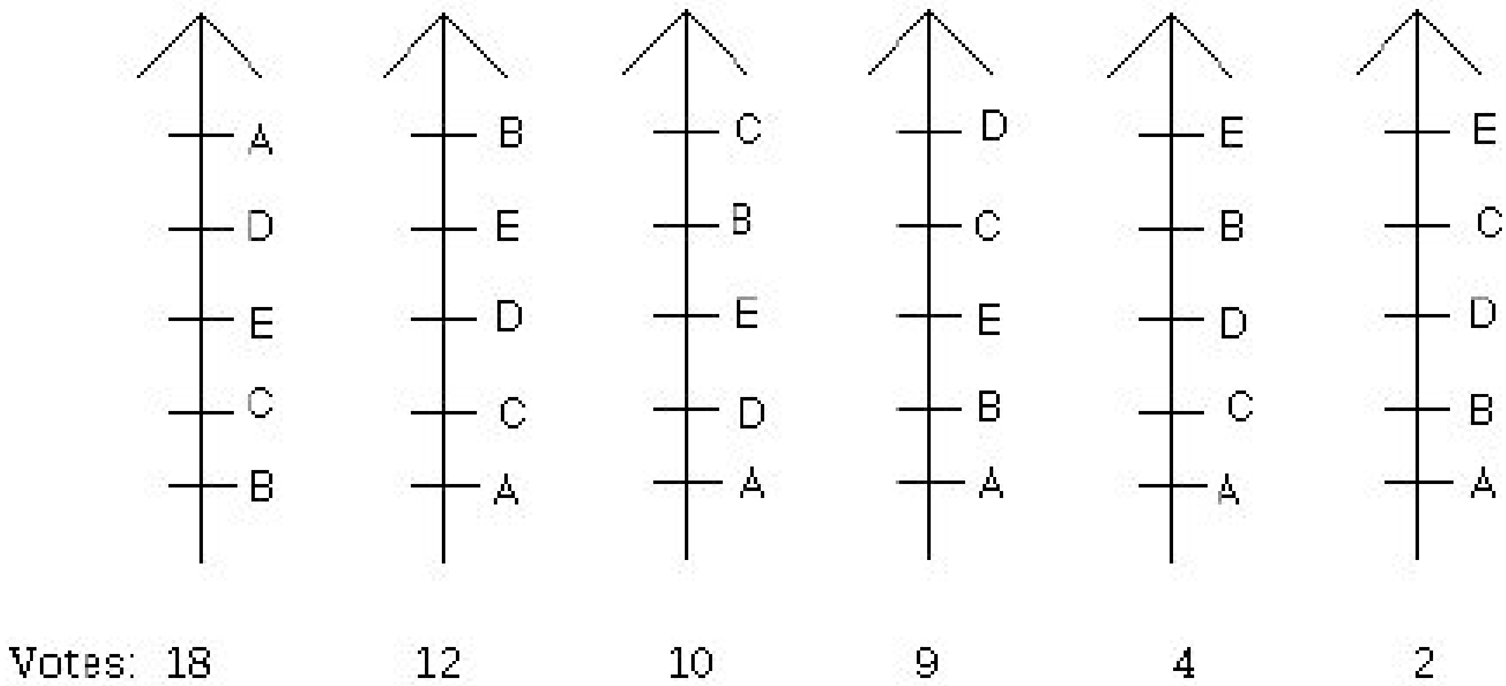
Notes for Remote Presentation 7:

Game Theory/Fairness Modeling

March 7, 2022



An election with different winners for different appealing methods to count the votes!

a. Plurality (largest number of first place votes)

b. Run-off (Run-off between the two candidates with the largest number of first place votes.

c. Sequential run-off (IRV)

If no candidate has a majority eliminate eliminate the candidate with the fewest first place votes and repeat the procedure

Another name for the system, especially when used for proportional representation, is the "single transferable vote."

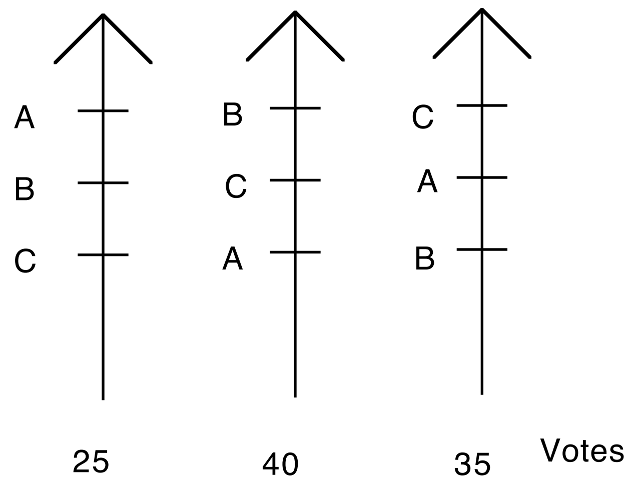
d. Condorcet (Winner is that person who can beat all others in a two-way race, assuming there is such a candidate.)

e. Borda count (Give each candidate i as many points on a ballot as there candidates below i on that ballot, and sum the points for the candidates. The candidate with the most points wins.

Why not turn the situation on its head and instead of looking for appealing ways to decide a winner based the ordinal ballots one looks for FAIRNESS PROPERTIES that a "good" election method should obey!

This suggests trying to chose among the different methods of deciding an election based on ranked ballots based on the APPEALING FAIRNESS properties they satisfy.

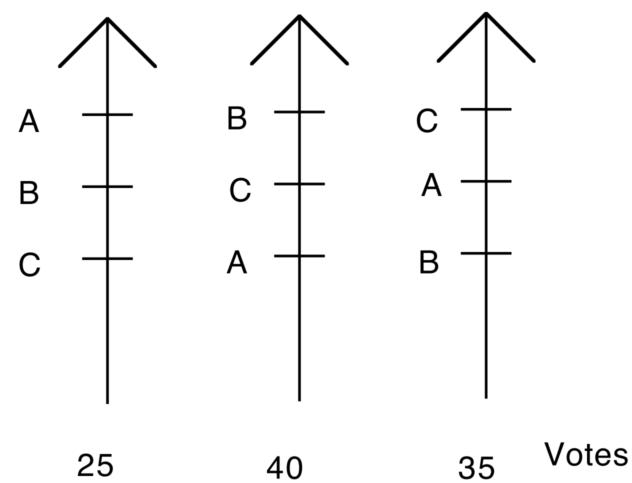
Independence of Irrelevant Alternatives and the Borda Count. (Note: No Condorcet winner.)



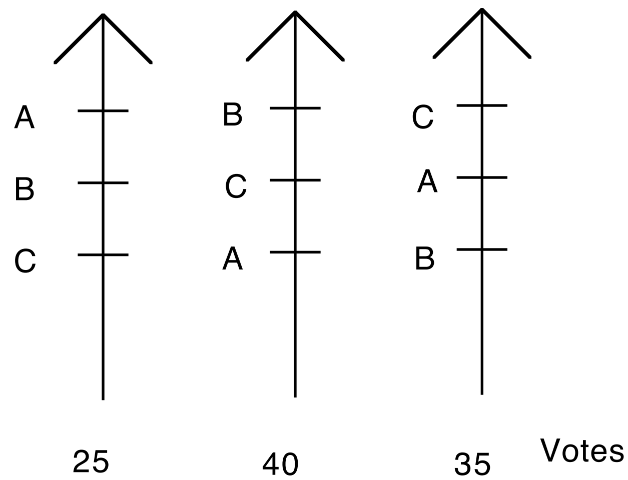
Borda counts: A gets 85; B gets 105; C gets 110 points.

B gets a higher Borda Count than A when C is present. But without C, 60 to 40 voters prefer A to B. So choice C is not irrelevant when using the Borda Count for the relative position of A and B.

Consequences of no Condorcet winner:



Note what happens if voting on items takes place sequentially in pairwise votes. Many real world legislatures work this way.



a. Vote on A vs. B; pit winner against C

A wins initially; C wins A vs C - C becomes law.

b. Vote on B vs. C; pit winner against A

B wins initially; A wins A vs B - A become law.

c. Vote on A vs. C; pit winner against B

C wins initially; B wins B vs. C - B becomes law

Thus, what becomes law depends on the *order* in which votes are taken!!

In a democracy one might want to avoid this but note that when opinion is sharply divided it is not clear how to go forward - legislatures may face this issue regularly.

This phenomenon is called cyclic majorities by political scientists and there seem to be examples where this kind of situation existed before the American Congress.

A famous example involved Adam Clayton Powell, Jr. (1908-1972) and federal aid to schools. The US Constitution leaves education to the states. Powell Amendment only wanted aid to ***desegregated*** schools!

Votes involved the three "choices" (candidates) were:

a. No Federal aid to public schools

b. Federal aid to all public schools

c. Federal aid to only desegregated public schools

Appealing methods that use ordinal (ranked ballots) as well as cardinal ballots (people give candidates a score on some scale, say from 0 to 9, 9 high) are being invented with regularity but all of them are subject to the "limitations" of Arrow's Impossibility Theorem and the Satterthwaite-Gibbard Theorem.

Another method which has actually been used.

Give each voter exactly 100 points to distribute as the voter wants to each of the candidates, and the candidate with the most points wins.

Sometimes this is called "cumulative" voting.

This is not range or score voting where each CANDIDATE is scored with points from 0 to 100 and the winner has the largest total.

Cumulative voting is sometimes used to elect a group of candidates rather than a single one.

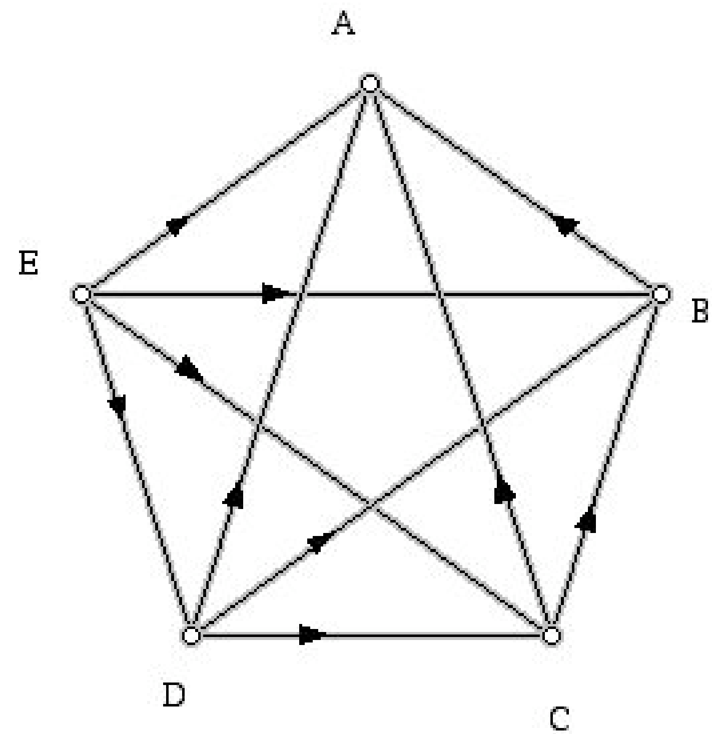
While third parties in America don't usually affect the outcomes of elections there are some examples where they may have had such an effect. Political scientists have data which suggest that different voting systems affect the "survival" or encourage the creation of parties that are relatively small compared with the "mainstream" parties.

Returning to our example:

Using tradition as a guide, the way nearly all American elections are decided, A would be the winner!

Even though a majority of voters liked A the least and A's percentage of first place votes is only about 33 percent A is the plurality winner.

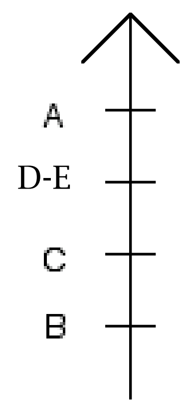
Here are the results of 2 way races: Perhaps you are surprised that E can beat each of the other candidates in a two-way race:



Perhaps the voting system should depend on "average" strength of the candidates with the voters.

Give points for how high up on a ballot the different candidates are.

This system is called the Borda Count, and the number of points for a candidate is the number of candidates below him/her.



A gets 4 points
B gets 0 points
C gets 1 point
D gets two point
E gets two points

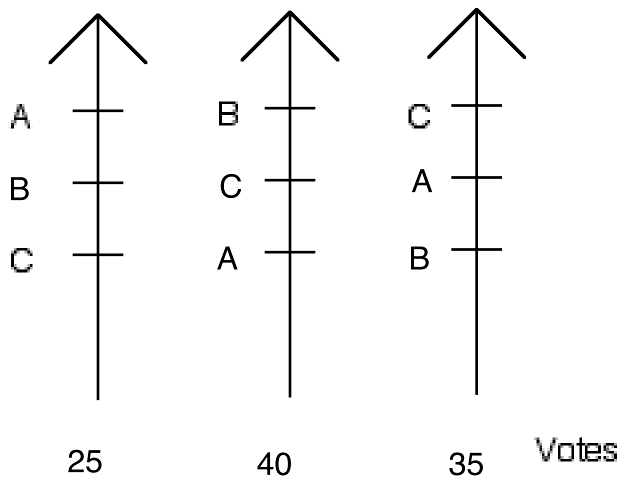
If 10 voters with this ballot, multiply by 10.

Nifty and perhaps unexpected way to get Borda Count results when there is no tie.

Construct a matrix showing votes for the alternatives (candidates) in two-way races.

Here is the data about the two-way races

The entry in row i and column j is the number of votes for choice i over choice j.



	A	B	C	
A	-	60	25	85
B	40	-	65	105
C	75	35	-	110

A's Borda count: 2(25) + 1(35) = 85
B's Borda count: 1(25) + 2(40) = 105
C's Borda count: 1(40) + 2(35) = 110

That the row sums of the pairwise preference matrix sum to the Borda Count is a theorem rather than an accident!

USA Presidential Election:

Direct election results:

a. ballot

b. decision method

Electoral college results:

a. weighted voting

b. weights are set by Huntington-Hill algorithm for the apportionment problem

To appear on the a state's ballot for election of the President a candidate must get the nomination of a party (the major ones are the Democratic and Republican parties) or get signed petitions from a large enough collection of voters.

The parties select nominees through a complicated process of primaries and caucuses. However, when a primary occurs, as shown by the recent (2016) Republican primaries, a candidate can "win" a state based on a small plurality of those who vote.

Arrow's Theorem

When there are 3 or more candidates no election method obeys a short list of fairness properties!

Kenneth Arrow majored in mathematics at City College and got his doctorate at Columbia.

Arrow's "fairness" conditions:

Voters produce ranked ballots with ties. One seeks a ranked ballot with ties for "society."

a. Procedure works for any ballots.
b. Non-dictatorial
c. Non-imposed
d. Monotone
e. (IIA) Independence of irrelevant choices

a. (Universal) No set of ballots is considered too "strange" to be considered for counting. The elections decision "committee" will not "edit/censure" a set of ballots.

b. No matter the election ballots society does not always agree with how a particular voter voted (no dictator).

c. Not imposed. If all voters vote for a particular candidate in first place that candidate should win.

d. Monotone - more support can't hurt; less support should not make one do better.

e. Independence of Irrelevant alternatives (IIA).

Relative position of X and Y should depend only on information about X and Y.

IIA has typically generated the most skepticism of the Arrow's axioms as regards its importance.

However, it is also tied up with with views about the interpersonal comparison of utility.

Gibbard-Satterthwaite Theorem

When there are three or more candidates all election methods except dictatorship can be "manipulated!"

Gibbard is a philosopher and Satterthwaite a management scientist.

The Borda Count seems like an appealing method though it does not obey Independence of Irrelevant Alternatives.

One obvious criticism: Suppose a voter knows the election will be close between his/her first and second place choices in an election when the voter provides honest preferences. By being honest, the voter's second place choice may "harm" his/her first place choice.

So a voter might be tempted to not vote honestly, but put his/her second place choice towards the bottom of his/her ballot to cut down on that candidate's points. However, all voters may vote dishonestly resulting in an outcome not at all in the interests of the group, or representative of their true opinions.

Weighted voting:

One person, one vote is an important notion in election theory for the United States. (Except the electoral college does not work that way!) More educated or wealthy Americans still only get one vote for who should be President.

How should one have representation in an amalgamation of countries of different sizes such the European Union or the counties of upstate New York where there are towns of very different sizes in a typical county?

County governments in New York provide funds for roads, county police, preventing river/lake contamination, etc.

Sometimes to be fair, different players should have different influence so voting is done using the idea of weighted voting.

Some players cast a bigger vote than others. Thus, each player i will cast a block of votes.

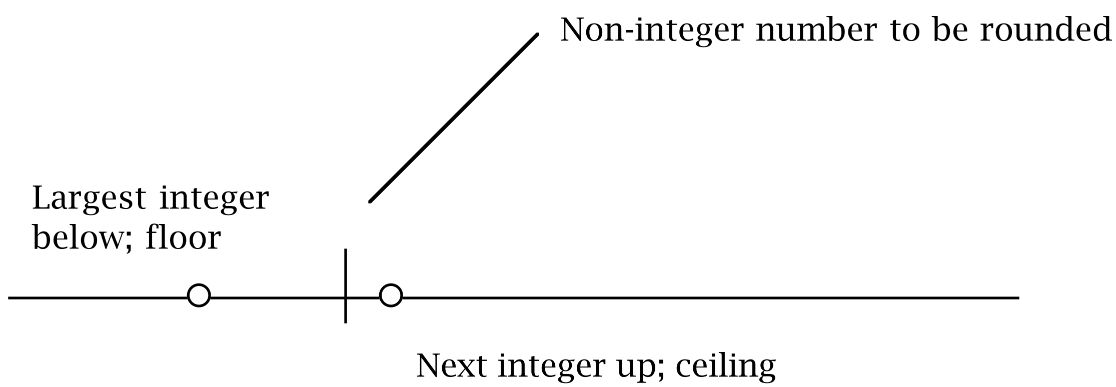
This leads to the idea of weighted voting. One has players 1, 2, 3,..., n each of whom casts a block of votes.

So player i casts w(i) votes, called i's weight.

A collection of players is called a *coalition*. To take action one needs to have sum of the weights of the players in a coalition exceed some number Q, called the Quota. Often the quota is set at the integer above 1/2(sum of weights) + 1

However, in some cases more than a simple majority is needed for action.

Rounding: Floor and ceiling functions: (Notation due to Donald Knuth, a computer scientist.)



$$\lceil 3.12 \rceil = 4 \ \text{(ceiling)}$$
$$\lfloor 5.88 \rfloor = 5 \ \text{(floor)}$$

Notation: players are "named" so that the small numbered players cast the largest votes (ties allowed).

Example:

[8; 6, 5, 2, 1]

Four players: player 1 casts 6 votes, player 2 casts 5 votes, player 3 casts 2 votes, and player 1 casts 1 vote.

[8; 6, 5, 2, 1]

For an action to be taken a group of "players" with at least weight 8 have to "work together." Thus, {1,2} is a coalition with players 1 and 2 and since 11 is bigger than 8, they can make sure that a "bill" passes the legislature. {1,2} is called a winning coalition.

[8; 6, 5, 2, 1]

{3,4} command only 3 points and so by themselves can't take action. This is a losing coalition.

A player is called a *veto* player if that player is a member of every *minimal winning* coalition.

That is, a group of players who if any player is deleted from that coalition can no longer take action. (Win)

Question: County Z has 5 towns with populations of 900,000, 500,000, 500,000, 400000, and 200,000. What might be a good set of weights and a quota for a county legislature with 5 players?

Weighted voting game:

[13; 9, 5, 5, 4, 2]

Minimal winning coalitions:

{1, 2} {1, 3} {1, 4} {2, 3, 4}

Player 5 has NO power! Player 5 is never a member of any MINIMAL winning coalition. That is, a group of players who if any player is deleted from that coalition can no longer take action (win). Players with positive weight who are never a member of a minimal winning coalition are called *dummies* in the weighted voting literature.

Example:

[5; 4, 3, 2] Three players named 1, 2, and 3 who cast 4, 3, and 2 votes respectively. The 5 is called the quota. Players with combined weight of 5 are needed to take action.

Is Player 1 twice as powerful as Player 3 because 4 is twice 2?

[5; 4, 3, 2]

Which coalitions (collections) of players can take action?

Minimal winning coalitions - no subset of a minimal winning (MW) coalition wins:

{1,2}, {1,3}, {2,3}

Given [5; 4, 3, 2], we have total symmetry here for the MW. The MW coalitions are:

{1,2}, {1,3}, {2,3}

so it should be apparent that in this game all three players have equal influence!!!

An isomorphic game would be:

[2; 1, 1, 1]

because its minimal winning coalitions are also:

$$\{1,2\}, \{1,3\}, \{2,3\}$$

Power indices: (Variants differ in using all winning versus MW coalitions) (Name are not standardized.)

a. Coleman

b. Banzhaf

c. Shapley

d. Deegan-Packel-Johnston

[5; 4, 3, 2]

MW: {1,2}, {1,3}, {2,3}

Coleman:

1 is in two coalitions
2 is in two coalitions
3 is in two coalitions

So 1 has 2/6 as a power; 2 has 2/6 as a power; 3 has 2/6 as a power!

Look at the pattern of Yes and No votes of the 3 players:

YYY wins
$\underline{\mathrm{YY}}$N wins
$\underline{\mathrm{Y}}$N$\underline{\mathrm{Y}}$ wins
YNN loses
N$\underline{\mathrm{YY}}$ wins
NYN loses
NNY loses
NNN loses

Underlines show when a Yes changed to a No changes a win to a loss. So of the underlined items each player has 2 out of a total of 6. (This is Banzhaf Power.)

So each player has equal Banzhaf power.

Note: We only look for "pivots/swing," that is changes when a sequence of Y's and N's wins, and changing a Y to an N makes a win a loss.

It turns out that looking at situations where a pattern yields a loss and changing a No to a Yes wins, just doubles the number of pivots/swings because we are computing a ratio.

Template for computing the Banzhaf power for a game with 4 players:

This pattern holds for n at least 2. n players 2^n lines in the table.

Pattern: Column 1, 8 Y's 8 N's; Column 2, 4 Y's, 4 N's, 4 Y's, 4 N's'; Column 3, Alternate 2 Y's, 2 N's, etc.

Y	Y	Y	Y
Y	Y	Y	N
Y	Y	N	Y
Y	Y	N	N
Y	N	Y	Y
Y	N	Y	N
Y	N	N	Y
Y	N	N	N
N	Y	Y	Y
N	Y	Y	N
N	Y	N	Y
N	Y	N	N
N	N	Y	Y
N	N	Y	N
N	N	N	Y
N	N	N	N

Shapley-Shubik Power Index

Consider all orders of the players voting. Reading from left to right give a pivot/swing point to the first player whose vote the sum of weights over the quota.

[5; 4, 3, 2]

1	*2*	3
1	*3*	2
2	*1*	3
2	*3*	1
3	*1*	2
3	*2*	1

Pivot player is shown in italics - second in every case for this example.

Hence:

Player 1 has 2 pivots out of 6; power 1/3

Player 2 has 2 pivots out of 6; power 1/3

Player 3 has 2 pivots out of 6; power 1/3

Remember that 2/6 is the same fraction as 1/3.

Banzhaf was not trained as a mathematician. He was trained in the law. He is most famous for winning cases against tobacco companies that smoking is harmful to one's health.

He also won a Supreme Court decision which overturned the use of weighted voting in Nassau County because there were players with NO Power!

Nassau and Suffolk now have legislatures rather than weighted voting but most upstate NY Counties have weighted voting procedures.

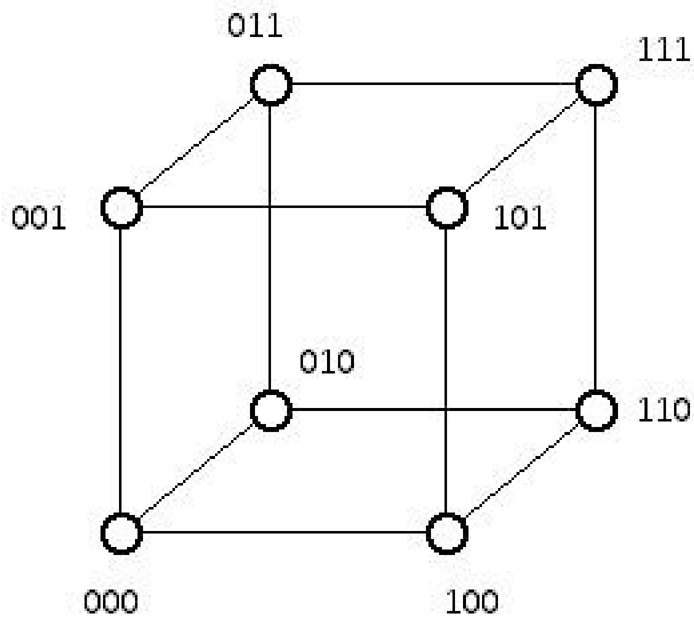
In NYS weights must be assigned to the players in the weighted voting games for county governments so that the Banzhaf Power is proportional to the population of the players involved.

Pattern of Yes/No for lines in a Banzhaf power table for 3-players "corresponds" to the labels needed for a 3-dimensional cube:

NNY,YNY,NYY,YYY(top)

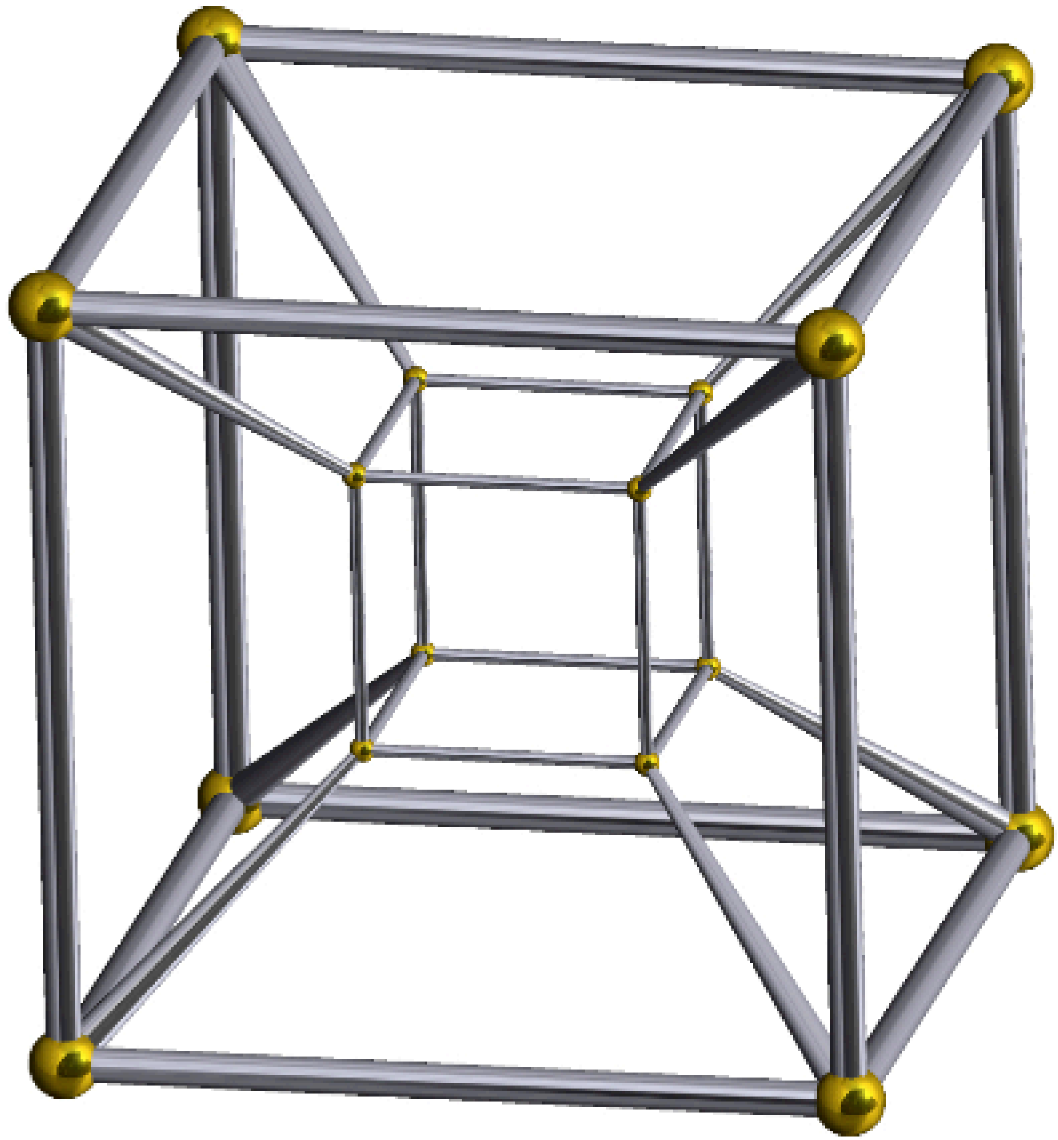
NNN,YNN,NYN,YYN(bottom)

Think of N as a 0 and Y as a 1:



Three-cube made from two 2-cubes! Top layer all entries end in 1; bottom layer all entries end in 0!

Banzhaf table for 4 players correspond is obtained by pasting together two copies of a 3-cube to get a combinatorial 4-cube.



One can look at power indices at the design phase of assigning weights in a weighted voting game.

But after the weighted voting "legislature" comes into existence once can look at the what one sees as HAVING HAPPENED in the real world with regard to the power relations.

At the design phase one might not filter in that two players will always vote the same way, but one see on an empirical basis that this is true.

For example one can see the Supreme Court as a kind of weighted weighed voting game with each player have one vote:

$$[5; 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

However, different historical versions of the Supreme Court often saw particular justices voting as a block. Sometimes such blocks depended on the kind of case before the Court.

For example of "civil liberties" cases there might be one pattern in which justices voted together but in "economic case" there might be another pattern.

The Coleman, Banzhaf, and Shapley-Shubik indices treat coalitions of different sizes the same way.

It is plausible that smaller coalitions are more likely to form so it might be reasonable to have a power index where there is a larger contribution to a player for being part of a small coalition as compared to a larger one. This is the idea behind the Deegan-Packel-Johnston index.

Setting the stage for learning about apportionment.

Classic problem: How many seats does each state in the US get in the House of Representatives based on its population or how many seats in a European parliament does each PARTY get based on the votes received by that party?

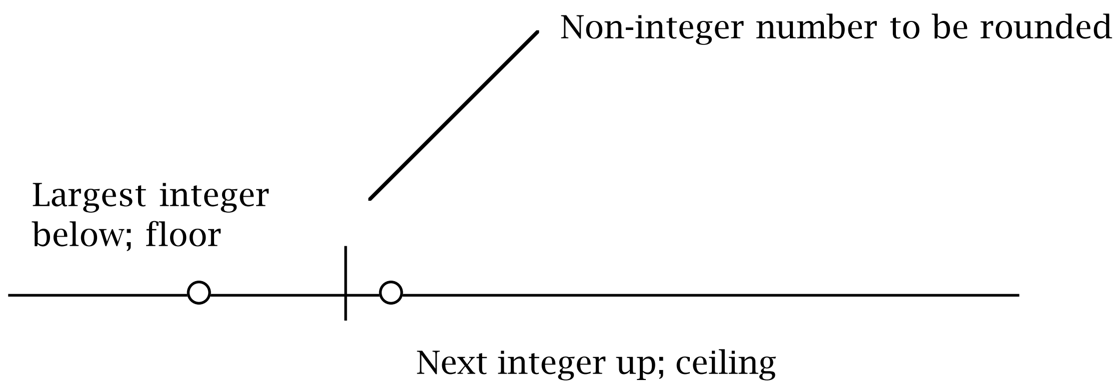
One strange feature of the US version of the problem:

Every state gets one seat regardless of how "undeserving" it might be.

In Europe, parties that get a very small percentage of the vote may get no seat!

Notation and its power:

Rounding:



Notation for floor and ceiling due to Donald Knuth

Floor function:

Largest integer smaller than or equal to x:

$$\lfloor x \rfloor$$

Ceiling function:

Smallest integer greater than or equal to x:

$$\lceil x \rceil$$

Examples:

$$\lceil 3.01 \rceil = 4$$

$$\lfloor 6.72 \rfloor = 6$$

Graph: $y = \lceil x^2 \rceil - (\lfloor x \rfloor)^2$

Practice:

$$-\lfloor 6.72 \rfloor =$$

$$\lfloor -6.72 \rfloor =$$

$$\lceil -12.47 \rceil =$$

Lots of interesting graphing examples can be done with this notation in a precalculus class in college or an algebra course in high school.

Absolute versus relative change:

A's population goes from

50,000 to 60,000

B's population goes from

5,000,000 to 5,010,000

Both A and B went up the same amount in "absolute" terms: 10,000 people.

But, 10000/50000 is 20%

while 10000/5000000 is .2%

Have a good week!

Questions: email me at:

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and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html