Notes for Remote Presentation 8:

Game Theory/Fairness Modeling

March 21, 2022

[8; 5, 4, 2, 1]

is a 4 player game where the quota is more than a *simple majority* of the weight.

Minimal winning coalitions are:

{1,2} and {1,3, 4}

Player 1 is a veto player - to get anything done, Player 1 must approve of the action, but Player 1 is not a dictator in this game because to win Player 1 needs the support of other players.

Coleman:

{1,2} and {1,3, 4}

Player 1 - 2/5
Player 2 - 1/5
Player 3 - 1/5
Player 4 - 1/5

The denominator is 5 because the the total number of entries in winning coalitions is 5.

[8; 5, 4, 2, 1]

Sample of Banzhaf lines:

YNYY passes;

*Y*N*YY* each Y if changed to an N changes outcome to a "fail."

YYYN passes

*YY*YN only first and second Y if changed to an N changes outcome to a "fail."

[8; 5, 4, 2, 1]

Sample of Shapley lines:

13*2*4 pivot to player 2 in position 3
1*2*34 pivot to player 2 in position 2
432*1* pivot to player 1 in position 4
413*2* pivot to player 2 in position 4

13*4*2 pivot to player 4 in position 3
14*3*2 pivot to player 3 in position 3

Apportionment:

Setting the stage for learning about apportionment.

Classic problems: How many seats does each state in the US get in the House of Representatives based on its population or how many seats in a European parliament does each PARTY get based on the votes received by that party?

One strange feature of the US version of the problem:

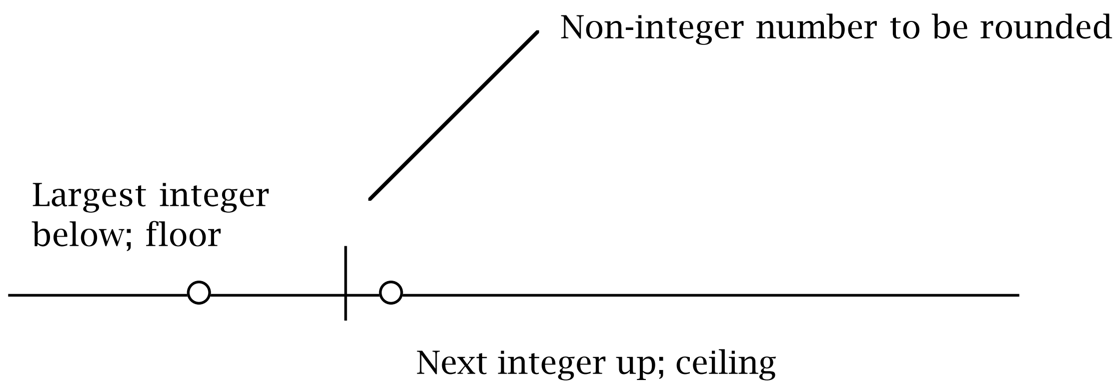
Every state gets one seat regardless of how "undeserving" it might be.

In Europe, parties that get a very small percentage of the vote may get no seat!!

If small parties in Europe were guaranteed seats then it might "pay" for parties to split up into smaller pieces to get additional representation!

Notation and its power:

Rounding:



Notation for floor and ceiling due to Donald Knuth

Floor function:

Largest integer smaller than or equal to x:

$$\lfloor x \rfloor$$

Ceiling function:

Smallest integer greater than or equal to x:

$$\lceil x \rceil$$

Examples:

$$\lceil 3.01 \rceil = 4$$

$$\lfloor 6.72 \rfloor = 6$$

Graph: $y = \lceil x^2 \rceil - (\lfloor x \rfloor)^2$

Practice:

$$-\lfloor 6.72 \rfloor =$$

$$\lfloor -6.72 \rfloor =$$

$$\lceil -12.47 \rceil =$$

Lots of interesting graphing examples can be done with this notation in a precalculus class in college or an algebra course in high school.

Absolute versus relative change:

A's population goes from

50,000 to 60,000

B's population goes from

5,000,000 to 5,010,000

Both A and B went up the same amount in "absolute" terms: 10,000 people.

But, 10000/50000 is 20%

while 10000/5000000 is .2%

Apportionment:

Here are two "not realistic" but instructive examples to try.

Three Claimants:

h = 12 (Size of the legislature)

A	B	C	
43	36	21	(Total 100)

How many seats for each claimant?

Four Claimants:

h = 12 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

What is the algorithm or method behind what you did so that you can do this for other examples with different numbers of claimants, population sizes and where h varies?

Four Claimants:

h = 12 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

Students typically discover for themselves (without any prior knowledge of this circle of ideas) two methods. Let us see how this might happen.

Some natural thoughts:

There being 12 seats ($h=12$) what is the exact share each claimed should get?

A	B	C	D	
38	27	23	12	(Total 100)

A has 38 percent of the population. Hence, .38(12) is A's fair share!

A is entitled to 4.56 seats but this is not an integer.

Fair share of all the claimants (which sum to 12)

A: 4.56
B: 3.24
C: 2.76
D: 1.44

Another view of fair share:

100 people are to be represented by 12 people:

$100/12 = 8.33333....$

So A's number of seats should be: 38/8.3333=4.56

B's share: 27/8.3333=3.24

C's share: 23/8.3333=2.76

D's share: 12/8.333 =1.44

By the laws of arithmetic (!) these are the same numbers we got before!

If claimant A's population is $P(A)$ and the total populations is T we have:

$$(P(A)/T)h = P(A)/(T/h)$$

Hamilton's Method:

Step 1:

We assign each claimant the integer part of their claim.

Step 2:

If these numbers don't add to 12, order the fractional parts in order of decreasing size and assign these in order of size until 12 seats are assigned.

In this example:

A: 4.56   B: 3.24   C: 2.76   D: 1.44

A gets 4 + 1 = 5 (over)
B gets 3 = 3 (under)
C gets 2 + 1 = 3 (over)
D gets 1 = 1 (under)

Total 12 as required.

Hamilton's method is easy to carry out and intuitive.

However, it has two big flaws:

Hamilton's Method fails "monotonicity." If can give out fewer seats to a claimant when the house size goes from h to h+1.

(known as the Alabama Paradox)

This is not serious in apportioning the House of Representatives because the number of seats stays 435.

Historical anomaly when Hawaii and Alaska became states.

More serious: Hamilton's Method is not population monotone. Various versions of this "axiom." Intuitively, a state could go up in population but get fewer seats.

This can happen with examples involving absolute growth of population or relative growth of population.

Another intuitive approach to dealing with the fractions: when fair share is not an integer, round using the usual rounding rule.

Fair share of all the claimants (which sum to 12)

A: 4.56	assign A 5 seats
B: 3.24	assign B 3 seats
C: 2.76	assign C 3 seats
D: 1.44	assign D 1 seat

Since we have assigned exactly 12 seats were are done.

Here when we us "usual rounding" we gave away exactly the right number of seats. But does this always happen?

Answer: NO!

Examples:

Four Claimants:

h = 13 (Size of the legislature)

A	B	C	D	
38	27	23	12	(Total 100)

How many seats for each claimant?

A's fair share: 4.94
B's fair share: 3.51
C's fair share: 2.99
D's fair share: 1.56

Giving A=5, B=4, C=3, D=2 apportions too many seats!

Four Claimants:

$h = 19$ (Size of the legislature)

A	B	C	D	
38	27	23	12	(Total 100)

How many seats for each claimant? (Quota: 5.26)

A's fair share: 7.22
B's fair share: 5.13
C's fair share: 4.37
D's fair share: 2.28

Giving A=7, B=5, C=4, D=2 apportions too few seats!

What is to be done?

Define EXACT QUOTA to be:

(Total population)/h

One can adjust this number up or down (to get an adjusted quota) depending on circumstances to assign exactly h seats after the rounding is carried out!

Four Claimants:

h = 19 (Size of the legislature)

A	B	C	D
38	27	23	12 (Total 100)

How many seats for each claimant?

If we divide by 5.1 instead of 5.26 we get:

A: 38/5.1=7.45 so 7

B: 27/5.1=5.29 so 5

C: 23/5.1 =4.51 so 5 (round up)

D: 12/5.1=2.35 so 2.

Total seats is 19 as needed!

Summary:

a: Hamilton's Method or the Method of Largest Remainders

Assign integer part of fair share and if more seats must be given out do this in order of the size of remainders.

b. Webster's Method (Sainte-Laguë)

Use ordinary rounding. If h seats are assigned, stop. Otherwise by trial and error adjust Exact Quota up or down to assign h seats.

Note: Technical detail: For the American version of the problem one has to sometimes modify any "algorithm" so that each state gets one seat as required by the Constitution!

Other important methods are:

Jefferson/D'Hondt

Huntington/Hill (used for the US to Apportion the House of Representatives)

Webster's Method is an example of a divisor method, based on usual rounding.

Other apportionment methods use a different rounding rule:

Jefferson (D'Hondt): Always round down

Adams: Always round up

Dean: Round based on the harmonic mean

Huntington/Hill (Currently the method used in America): Round using the geometry mean

Geometric mean of a and b equals square root (ab)

How does one decide which of these methods is better or worse?

What fairness properties do they obey?

Two major approaches:

a. Global optimization based on all 50 states.

Say, find number of seats for each state to minimize sum of absolute difference for total population/435, and state population/(mean district size).

b. Pairwise fairness between states (parties) - suppose state i gets s(i) seats and j gets s(j) when will it be "more fair" to switch one seat between states i and j?

What measure of "fairness" in comparing two states might we use?

$$|(pop(i)/s(i))-(pop(j)/s(j))|$$

people per district

$$|(s(i)/pop(i))-(s(j)/pop(j))|$$

seats per person = 1/(people per district)

Unintuitively, these two measures yield different solutions.

Next week we will analyze this!

Have a good week!

Questions: email me at:

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and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html