Notes for Presentation 2:

Game Theory/Fairness Modeling

January 31, 2022

Let us take a moment to see how a geometric tool can help with insight into game theory. It will be useful to you in many other contexts as well.

This topic belongs to the areas of mathematics known as discrete mathematics, combinatorics, discrete geometry and graph theory.

Dots and lines diagrams known as:

*graphs

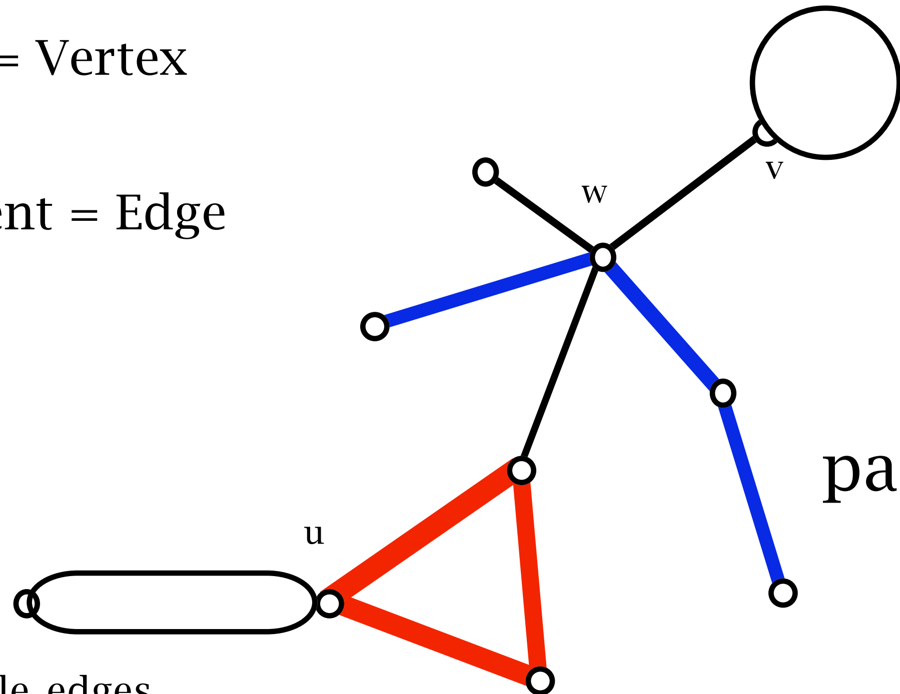
* digraphs (directed graphs) (arrows on the lines)

Introduction (primer) of graph theory:

(Think about how the difficulty of these ideas compare with solving equations! One "downside," lots of new words at the start.)

Dot = Vertex

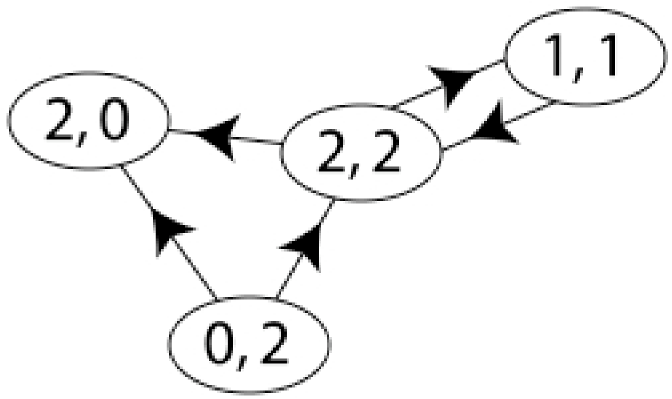
Line segment = Edge



This graph has 10 vertices and 12 edges.

The valence or degree of a vertex in a graph is the number of (local) line segments which meet at the vertex. The valence of v is 3, of w is 5, and of u is 4.

Digraph: 4 vertices; 5 directed edges.

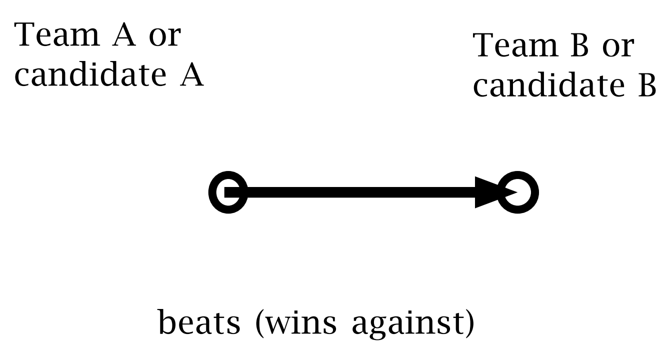


Numbers show indegree and outdegree of vertices.

Indegree: number of directed edges coming into a vertex

Outdegree: number of directed edges leaving a vertex

A major application of digraphs is to the analysis of the outcomes of sports and other tournaments. If team A beats B in a match (or in an election) one draws an arrow from A to B, and often one labels the arrow with the score. (A beats B by 7 to 3.)



Now let us return to thinking about how to play zero-sum games wisely.

Example of a zero-sum game with two actions for each of the two players, Row and Column:

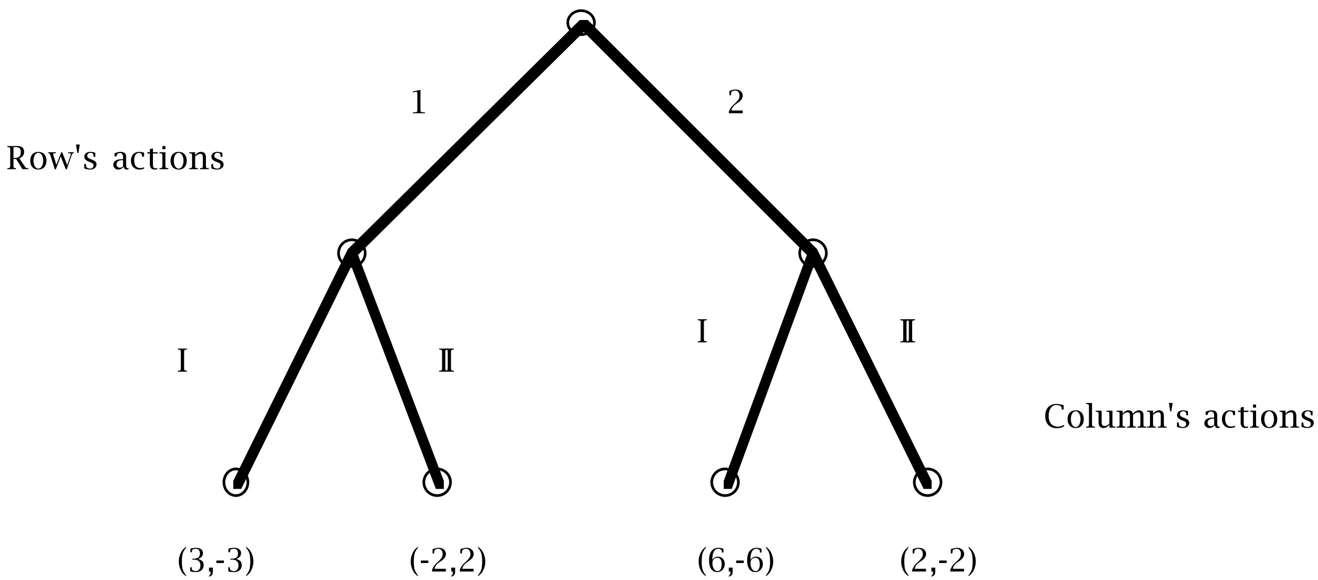
Moves for player named Column

Moves for player named Row.

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(6, -6)	(2, -2)

A more flexible way to represent games than matrices is to use tree diagrams.

Tree of a matrix game:



Outcomes at the leaves are payoffs, Row's is the first number; Column's is the second number.

How might Row and Column reason about how to play such a game?

Moves for player named Column

Moves for player named Row.

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(6, -6)	(2, -2)

For Row: whatever Column does, row 2 is better than row 1! (6>3; 2>-2)

Moves for player named Column

Moves for player named Row.

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(6, -6)	(2, -2)

For Column: whatever Row does, column II is better than column 1! (2>-3; -2>-6)

So it makes sense, for no matter how many independent plays are made of this game, for Row to always play row 2 and Column to always play column II. Payoff every time is: Row wins 2, Column loses 2. The game is UNFAIR: Row always wins, Column always loses when both play OPTIMALLY!

If the players of the original game are playing "rationally," it is as if they were playing the following 1x1 matrix game:

	Column II
Row 2	(2, -2)

A dull game to play especially for Column, who always loses.

Note that Row moves by picking a row to "play." Column moves by picking a column to play.

When one finds a row that dominates another row one can get a SMALLER game matrix by CROSSING out the row which is DOMINATED, leaving the dominating row intact.

So a first step in analyzing how to play a zero-sum matrix game is by looking for rows or columns that might dominant other rows or columns.

Note: Initially there may be NO dominating row but after eliminating a dominated column, there may be a dominating row.

Initially there may be NO dominating column but after eliminating a dominated row, there may be a dominating column.

Thus, Row (the row player) looks for dominating rows.

Thus, Column (the column player) looks for dominating columns.

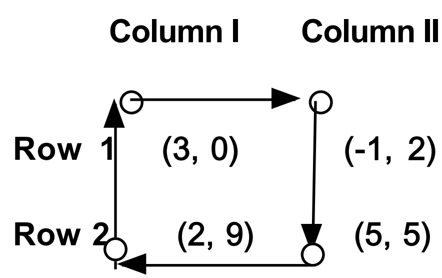
Simplify this **zero-sum** game matrix as much as possible. Payoffs are from Row's point of view. A payoff of -4 is a GAIN for Column and a loss for Row.

	I	II	III
1	2	0	-9
2	3	1	-7
3	1	-4	2

What does one do if dominating strategy analysis does not simplify the game matrix of a zero-sum game?

Here is a tool to help one think about such games better, and uses a digraph.

Example: Motion diagram of a 2x2 matrix game - in this case the payoffs are *not* zero-sum:



One dot for each payoff.

This diagram shows that there is no outcome that is STABLE because one of the players has an incentive to change his/her actions. A "stable" outcome would be one with OUTDEGREE zero.

This game has no dominating rows or columns: How would you play this game zero-sum?

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

Suppose you were required to play this game 100 times - a new round after each prior round is completed.

How would you decide what sequence of moves to make?

Suppose you are Column and you notice that Row always plays this pattern of rows:

1, 1, 2, 1, 1, 2, 1, 1, 2, ...

What would you do?

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

Row's pattern:

Rows: 1, 1, 2, 1, 1, 2, ...

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

If Row said I plan to play Row one for sure, what would Column do?

Column can be sure of getting a positive payoff by playing column II!

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

If Row says I plan to play Row 2 for sure, then Column can be sure to win by playing column I.

Thus, if either player detects a pattern in the play of their opponent, they can exploit that information to get BETTER outcomes!

Thus, for this game:

	Column I	Column II
Row 1	(3, -3)	(-2, 2)
Row 2	(-1, 1)	(7, -7)

optimal play for both players requires they use of a randomization device!!!

Fact:

For many zero-sum games that are played repeatedly, the best way to play is NOT deterministic but using some way to make one's choices randomly!

But there are infinitely many ways of playing randomly. Is there one which is best?

Answer: Yes! We will see how to find that BEST way of playing randomly.

Review of randomization and probability theory!

When one does an "experiment" there is a collection of possible outcomes, assumed to be finite for our purposes.

Example: Toss a coin:

Outcomes: head or tail

Example: Roll a standard die (plural is dice)

Outcome: 1 to 6 "spots"

Some coins are "fair" and some are "biased."

Some dice are "fair" and some are biased.