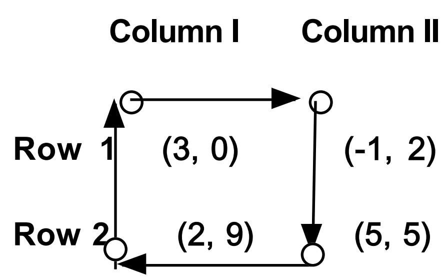
Session 3

Game Theory/ Fairness

Feb. 7, 2022

Here is a tool to help one think about such games better, and uses a digraph.

Example: Motion diagram of a 2x2 matrix game - in this case the payoffs are *not* zero-sum:



One dot for each payoff.

This diagram shows that there is no outcome that is STABLE because one of the players has an incentive to change his/her actions. A "stable" outcome would be one with OUTDEGREE zero.

Motion diagrams can be used to look at the "stability" or "equilibrium" behavior of nxm zero-sum or non-zero-sum matrix games.

Some ideas related to probability theory and the outcomes of an "experiment."

Example: Toss a coin:

Outcomes: head or tail

Example: Roll a standard die (plural is dice)

Outcome: 1 to 6 "spots"

Some coins are "fair" and some are "biased."

Some dice are "fair" and some are biased.

What pattern of heads and tails does one see when one tosses a fair coin 20 times?

A fair coin will show heads and tails in approximately equal frequencies. Thus if one tosses a FAIR coin 20 times, one will get approximately 10 heads and approximately 10 tails, the sum being 20.

However, and this is a sticking point for most beginners, when one tosses a fair coin 20 times the chance of getting exactly 10 heads and 10 tails is very low!!

In fact the chance of this happening is:

.0000976525

About one in ten thousand experiments!

Given the set S of n outcomes of an experiment, called the sample space of the experiment:

$$S = \{ o_1, o_2, o_3, \ldots, o_n \}$$

Probabilities of events obey:

1. $p(o_i) \geq 0$ (probability of each event is positive or 0.)

2. $p(o_i) \leq 1$ (probability of each event is at most 1.)

3. The sum of the probabilities of all the sample space events adds to 1.

Note: The standard symbol for summation in mathematics is the Greek letter "capital" sigma:

$$\Sigma$$

Thus:

$$\sum p(o_i) = 1 \text{ (summed from } i = 1 \text{ to } n)$$

How can one "interpret" probability in the "real world?"

a. Long term relative frequency

b. Intensity of a "belief."

We need to understand the difference between the probability of something happening, and the EXPECTED value of an event or outcome.

Expected value is the probabilistic analogue of finding the mean of a collection of numbers:

Mean of: 1, 3, 2, 3, 5

$(1+3+2+3+5)/5 = 14/5 = 2.8$

Note: the mean is not one of the ORIGINAL data values!

A ***fair*** coin is tossed three times. What is the expected number of heads?

Outcomes:	Number heads:
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2
HHT	2
HHH	3

Mean: Sum/8 = 12/8 =1.5

Note: The mean values is not a possible outcome which can only be 0, 1, 2, or 3.

Binomial Model:

For the general situation where there are two outcomes: A and A' (not A, the complement of set A), from an experiment repeated n times.

The expected number of times A occurs will be:

$$p(A)n = p(A) \times n$$

The number of times A' occurs is: $p(A') \times n$

Example: a fair coin is tossed 21 times:

expected number of heads is:

$(1/2) \text{x} 21 = 10.5$

Theorem: If we have numerical outcomes (random variable) $x(i)$ where $p(x(i))$ is the probability of $x(i)$ occurring the expected value is given by: Sum over all outcomes of:

$$\text{sum: } x(i)p(x(i))$$

For the biased coin tossed 3 times:

$p(H)=1/4$ and $p(T) = 3/4$

$n \times p(H)$ gives: $3(1/4) = 3/4$

$n \times p(T)$ gives: $3(3/4) = 9/4$

Note $3/4 + 9/4 = 12/4 = 3$
(There were 3 tosses.)

Expected number of heads:

Details of the calculation: (sum product of outcome and its probability)

TTT	0	27/64
TTH	1	9/64
THT	1	9/64
THH	2	3/64
HTT	1	9/64
HTH	2	3/64
HHT	2	3/64
HHH	3	1/64

Expected value for the number of heads is:

$(0+9+9+6+9+6+6+3)/64 = 48/64 = 3/4$

Fundamental principle of counting:

If I have 4 shirts and 5 pairs of jeans I can wear 20 different outfits?

When choices are made independently, we multiply the choices to get the number of outcomes:

m choices; n choices

Total number of possibilities: mxn

This is called the multiplication rule.

If a toss a coin followed by rolling a die followed by rolling another die there are:

2x6x6 = 72 possible outcomes!

Multiplication rule for probabilities:

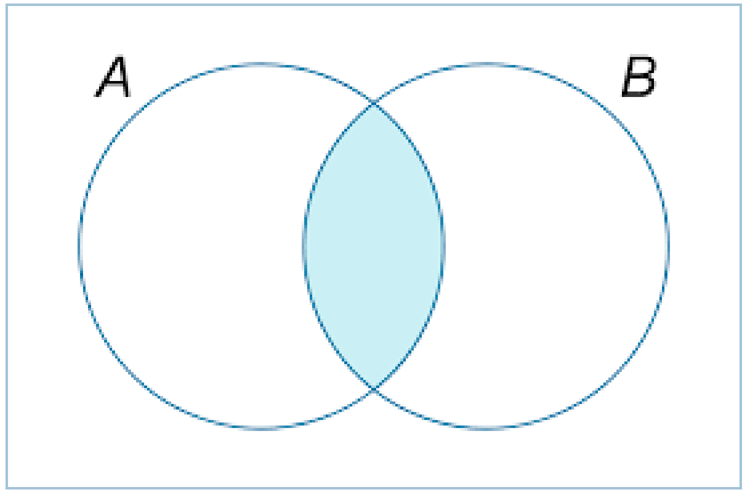
If events are "independent"- intuitively don't affect one-another where:

Event A: probability p(A)

Event B: probability p(B)

$$p(A \text{ and } B) = p(A \cap B) = p(A) \times p(B)$$

We will come back to the notion of conditional probability.



Outcomes where both A and B occur are shown blue. The outcomes in one or both of the circles is called A union B. $A \cup B$. (A occurs or B occurs or both occur.)

A fair biased coin p(head) = 1/4 is tossed followed by rolling a fair die. What is the probability of getting a tail and a 6?

Since the p(head) = 1/4, p(tail) = 1-1/4

p(tail) = 3/4; p(6) = 1/6

Probability of a tail and a 6 is:

(3/4)(1/6) = 3/24 = 1/8

		1/5	4/5
		Column I	Column II
1/4	Row 1	3	-1
3/4	Row 2	-9	7

$$p(\text{outcome } 3) = (1/4)(1/5) = 1/20.$$

$$p(\text{outcome } 7) = (3/4)(4/5)=12/20$$

$$p(\text{outcome } -1) = (1/4)(4/5)=4/20$$

$$p(\text{outcome } -9) = (3/4)(1/5)=3/20$$

Note: These numbers account for all outcomes so add to 1.

If the players use these "spinners," what is the payoff to Row?

Row earns:

$$(1/20)\text{x}3 + (12/20)\text{x}7 - 1(4/20) - 9(3/20) =$$

$$(3 + 84 - 4 - 27)/20 = 56/20 = 2.8$$

On each play of the game Row wins on average 2.8 and Column loses 2.8

Can Column do better?

How should one play this game? Is it fair?

	Column I	Column II
Row 1	3	-1
Row 2	-9	7

Optimal play means designing a "spinner" for each player that gives the best outcome for each player, even if it means that one player has an advantage and the game is not fair.

Row plays Row 1, p percent of the time. Column plays Column I, q percent of the time.

		q	1-q
		Column I	Column II
p	Row 1	3	-1
1-p	Row 2	-9	7

Note: (3 + (-1) + (-9) + 7) = 0. **Does this mean the game is fair? (No)**

		q	1-q
		Column I	Column II
p	Row 1	3	-1
1-p	Row 2	-9	7

Value to row: EV =

$$3pq + (-1)(p)(1-q) + (1-p)(q)(-9) + (1-p)(1-q)(7) =$$

$$3pq + pq + 9pq + 7pq - p - 7p - 9q - 7q + 7 =$$

$$20pq - 8p - 16q + 7 = 20p(q - 8/20) - 16q + 7 =$$

		q	1-q
		Column I	Column II
p	Row 1	3	-1
1-p	Row 2	-9	7

$$= 20p(q-8/20) -16q + 7 =$$

$$20p(q-8/20) - 16(q-8/20) +7 - 16(8/20) =$$

$$(20p -16)(q - 8/20) + (140/20) - (128/20) =$$

$$= \quad \mathbf{20(p-4/5)(q-2/5) + 3/5}$$

What have we learned?

EV(Row's viewpoint) = 20(p-4/5)(q-2/5) + 3/5

Row should play Row 1, 4/5 of the time.
Column should play Column I, 2/5 of the time.

The game is not fair: Row win 3/5 on average with every play of the game. Column "gains" -3/5 (loss)

Amazing fact: When Row plays Row 1, 4/5 of the time it does not matter what Column does because the first term above is 0!! (Any number times 0 is 0!) When Column plays Column I, 2/5 of the time it doesn't matter what Row does. Column gets the best outcome possible against a "rational" Row player.

General Theorem: (Solution of the game.)

For a 2x2 game zero-sum game with no stable point (motion diagram is a cycle) optimal play requires using randomization and player payoffs are governed by an expression of the form:

$$EV = C(p-a)((q-b) + K$$

where p and q are the percentage of the time that Row plays Row 1 and q is the percentage of the time that Column plays Column 1. The game is fair when K = 0, otherwise it is not.

Using the theorem to simplify calculations:

If Row want to design an optimal spinner for a 2x2 game with no stable point, Row knows that the payoff gotten will be the same if Column always plays Col I or always plays Column II. We can use this to find the value of p for Row's optimal spinner.

		q	1-q
		Column I	Column II
p	Row 1	3	-1
1-p	Row 2	-9	7

Thus:

3p -9(1-p) = -p + 7(1-p) (*)
12p-9 = -8p + 7
20p=16
p=4/5 (the same value we got from the earlier more complex calculation. EV(row) = 3(4/5)-9(1/5) = 3/5 (or -(4/5) + 7(1/5) = 3/5). Substitute 4/5 in one side of (*))

(This confirms the earlier result).

What can one say about zero-sum games with more than 2 rows and more than 2 columns?

Worst action for Row

Worst action for Column

Best action for Row

Best action for Column

No dominating row or column:

Row/Column	I	II	III
1	-3	2	-4
2	2	-6	-5
3	-7	3	-6

(Row 1, Col III) is a stable point. Minimum in its row and maximum in its column.

This choice is optimal. "Nash equilibrium."

To get further insight we have to look at games which are not zero-sum:

	Column I	Column II
Row 1	(3,3)	(-4,4)
Row 2	(4,-4)	(-1,-1)

What is rational play in this game? Answer next week!

Mathematical model:

A simplification of a situation in the "real world" so that one can carry out an analysis.

Mathematical models can be physical or "conceptual."

Example of a physical model:

Give examples where voting and elections are used in our lives:

What are the features (components) of an election or voting system that we are trying to understand elections so we can "improve" the way the election is carried out?

Components of an election or voting system:

1. One needs voters or decision makers.

Say n voters or decision makers.

2. One needs alternatives or candidates to vote on or choose from.

Say there are m candidates.

3. One needs a way for the voters to express their opinions about the choices or candidates.

The usual way this is done is by using a ***ballot***. We also need to think about how voters behave in filling out ballets.

4. Based on the ballots one needs a way of deciding who the winner or collection of winners is. Sometimes one is filling seats on a committee and there may be several people elected.

What types of ballots are you familiar with from elections in which you have participated?

In America we vote for:

* President

* Members of the House of Representative and Senate

* Governors

* Mayors

* Chairperson of a department

* Faculty committees

* Best actress

* Best movie

* Best rookie pitcher

* Best player in a particular football game

Mathematics has explored the surprisingly many ways to to construct ballots as inputs to elections.

The the major distinction parallels the two major kinds of numbers we use:

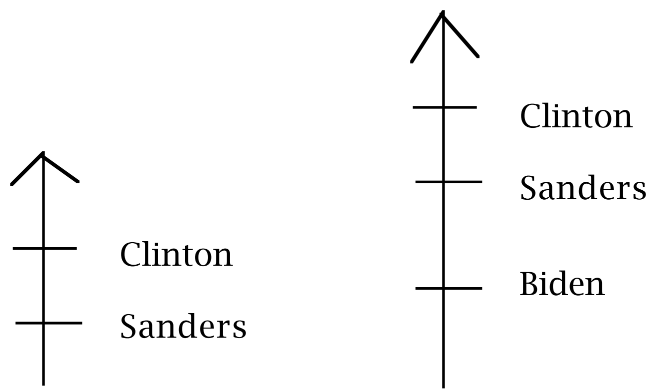
ordinal (counting numbers)

cardinal numbers (to measure)

* ordinal or rank ballots (with or without ties)

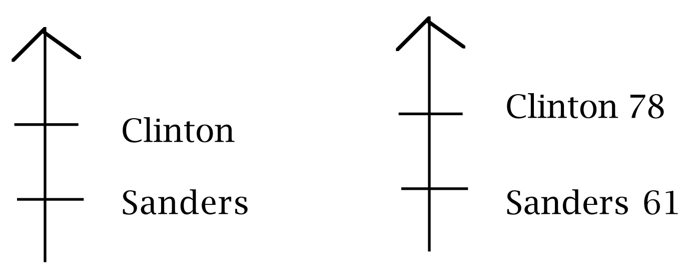
Show order of the candidates (choices) but not how strongly one feels about the candidates.

Two and three candidate ordinal ballots:



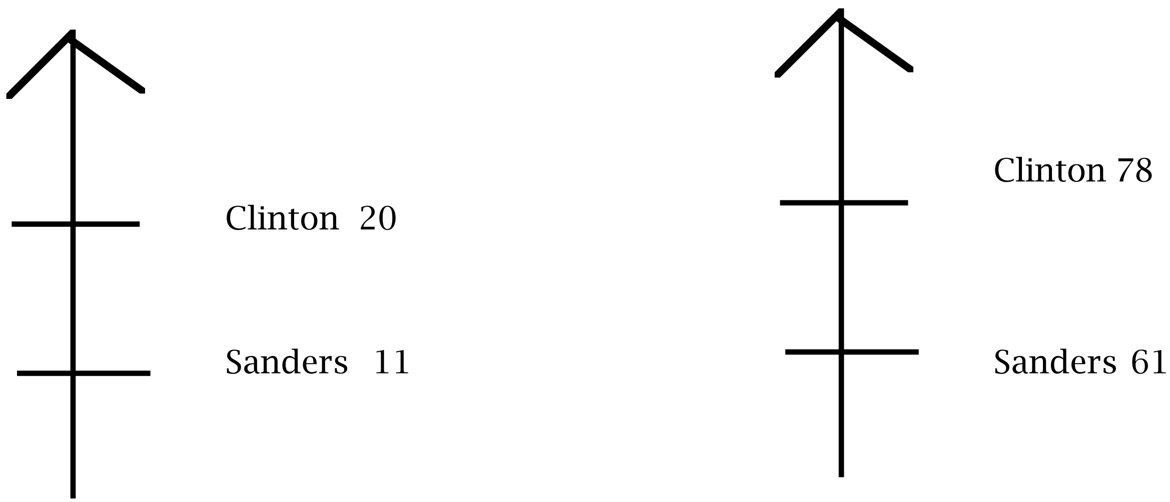
* cardinal ballots show intensity of support

Scale (100 high; zero low)



Same ranking but very different information about "intensity."

Scale: 100 high; 0 low



Is ballet truncation allowed?

Truncation on a ballot refers to the voter not listing all of the candidates but only some of the candidates.

Truncation can occur because the voter chooses to not list candidate he/she knows or the voter may not know anything about the candidate.

Sometimes a ballot is truncated because the voter knows what method is being used to count the ballots, and voting for more than one candidate will help not only one's favorite but other candidates as well. This called ***strategic voting***.

Such voting is called strategic. One "lies" about one's true views about the candidates to help a particular candidate or group of candidates win. Thus, one might only rank "conservative" candidates in a primary election with many people seeking office.

In some elections TRUNCATION is not permitted! Sometimes votes must rank the candidates without ties.

Have an enjoyable week!

If you have questions email me:

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class web page:

https://york.cuny.edu/~malk/gametheory/index.html