Comparison of Election Methods Regarding Winners and Rankings (2022)

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Plurality voting is widely used, especially in English speaking countries despite the fact that when there are three or more choices to be voted on this system often leads to "unsatisfactory" results. Here "unsatisfactory" is being used in the sense that some other outcome would have had more "support" from the voters.

I will describe some issues related to how plurality voting stacks up against such other systems as Condorcet and the Borda Count, which will be described briefly below, using an example reported on by Donald Saari. Figure 1 shows the results of an election involving three choices where 15 voters participated. In this diagram choices towards the top are more preferred but this notation allows for voters listing two choices at the same level, that is, have a tied preference. Also, notice that using ballots such as those in Figure 1 rather than a ballot where only one's favorite choice is asked for, is rare in "real world" elections. Note that starting in 2021 New York City will us ranked ballots (ordinal ballots) for local elections. However, the additional expressiveness that this ballot allows in voter choice does not prevent one from choosing a winner based on "first-place votes." Here A gets 6 first-place votes, B gets 5 first-place votes, and C gets 4 first-place votes. So using plurality voting, or sometimes it is called first past the post, the winner of the election is A. However, we might also care about the order of the choices based on plurality votes. This is called the plurality ranking. Figure 2 shows this ranking.

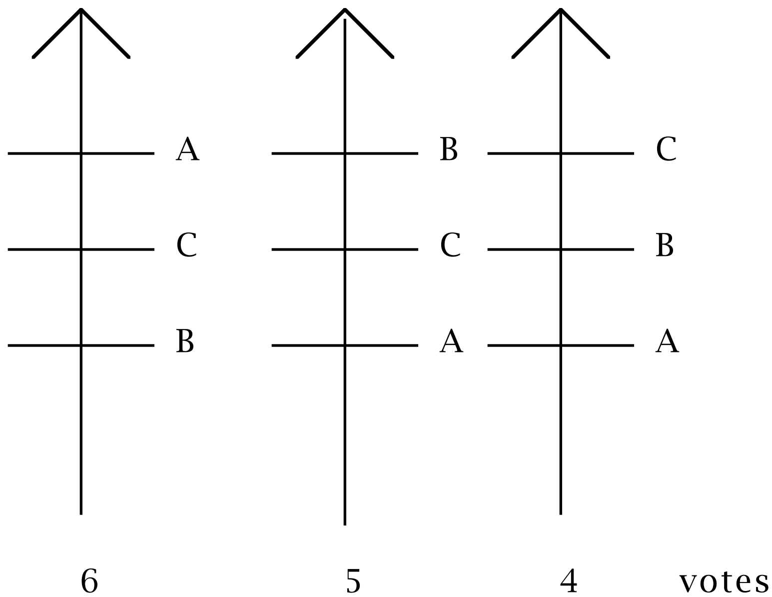

Figure 1 (An election where 15 voters provide ordinal ballots. Higher choices towards the top.)

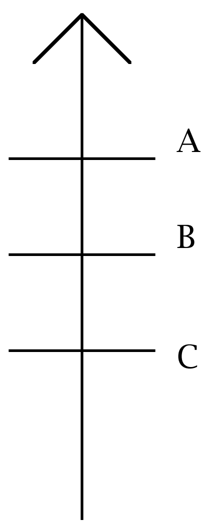

Figure 2 (Plurality ranking for the election in Figure 1.)

To get better insight into the voter's "feelings"/views about three candidates based on the ballots in Figure 1, we might construct the following matrix (table) (Figure 3) where the entry in row i column j (when i is not equal to j) gives the number of ballots where candidate i was preferred to candidate j. Thus, the entry in row 3 column 1 of 9 indicates that there were 9 ballots where candidate C was preferred to candidate A.

	A	B	C	Row sum
A		6	6	12
B	9		5	14
C	9	10		19
Column sum	18	16	11	

Figure 3 (Pairwise votes for the election in Figure 1.)

Two different approaches to using ballots to provide a winner or ranking of the alternatives that voters express opinions about are due to Borda (1733-1799) and Condorcet (1743-1794). The intuition behind the Borda Count is that one should try to find some way to see how well each candidate did on the ballots on average (mean). This can be done by giving points to candidates based on how high up they appear on the ballots. One approach, which works not only for the ballots in Figure 1 but more generally for ballots of this kind where ties are allowed, is to assign points based on the number of candidates below a given candidate on a ballot. Thus for the 4 ballots where C is ranked highest we would give C 2 points, B 1 point and A 0 points. Using this approach here are the point scores for the 3 candidates:

A gets: 2(6) + 0(5) + 0(4) = 12

B gets: 0(6) + 2(5) + 1(4) = 14

C gets: 1(6) + 1(5) + 2(4) = 19

So C wins the election using the Borda Count and the Borda ranking of the three candidates is shown in Figure 4. Note that the three choices appear in the reverse order from that of the plurality ranking (Figure 2). Also note that the Borda Counts we computed directly are equal to the row sums of the matrix in Figure 3. This is not an accident and is true in general, that is, one can prove a theorem to this effect. You may wonder about, and think about, if the column sums can be given a "natural" meaning.

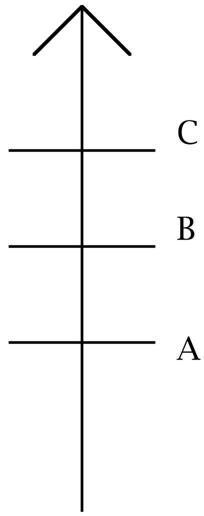

Figure 4 (Borda and Condorcet ranking for the Figure 1 election.)

Another appealing method of trying to rank or decide the winner when people use ordinal or ranked ballots (as in Figure 1) to vote is known as the Condorcet method. A Condorcet winner in an election is a candidate who can beat all the other candidates in a two-way race. A Condorcet loser in an election is someone who is beaten by all of the other candidates in a 2-way race. For the election in Figure 1, Figure 5 shows the two-way races between pairs of candidates. The arrow from B to A means choice B beats choice A in a two-way race. You can use the table in Figure 3 to verify the results.

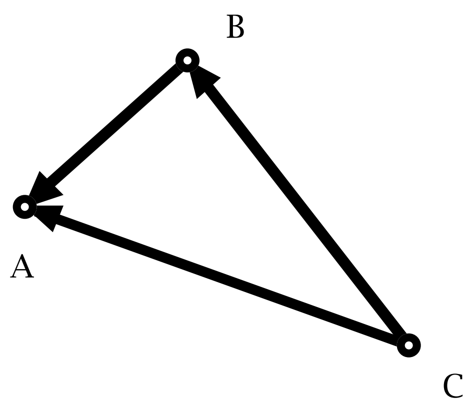

Figure 5 (Pairwise wins for the election in Figure 1.)

In this case C is the Condorcet winner and A is the Condorcet loser. And the ranking generated by this method is shown in Figure 4; the Condorcet ranking is exactly the same as the Borda ranking though this is not always the case.

Saari calls attention to the fact that a Condorcet winner can never be at the bottom of a Borda count ranking, and a Condorcet loser cannot be top ranked by the Borda Count. The first observation is related to the fact that Baldwin's method, which selects a winner based on eliminating the candidate with the lowest Borda count over and over again until a winner emerges, who, if there is a Condorcet winner, must be the Condorcet winner. Note that the Borda winner can be different from the Baldwin winner.

A natural question to ask is if there is a best method of electing a winner when ordinal ballots are used to rank candidates? This question has been studied in great detail, catalyzed by the work of the late Kenneth Arrow who shared the Nobel Memorial Prize in Economics (with another "mathematical" economist) in part for his work addressing this question. Over simplifying Arrow showed if that there are three or more choices to be made based on ordinal ballots (indifference-ties between choices is allowed) produced by a collection of voters, there is not a "perfect" method (in the sense of obeying a reasonable collection of fairness axioms or rules) to produce a ranking of the choices, allowing ties.

References:

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